

# Dispersive and Strichartz estimates for hyperbolic equations with constant coefficients

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## Abstract

Dispersive and Strichartz estimates for solutions to general strictly hyperbolic partial differential equations with constant coefficients are considered. The global time decay estimates of  $L^p - L^q$  norms of propagators is discussed, and it is shown how the time decay rates depend on the geometry of the problem. The frequency space is separated in several zones each giving a certain decay rate. Geometric conditions on characteristics responsible for the particular decay are investigated. Thus, a comprehensive analysis is carried out for strictly hyperbolic equations of high orders with lower order terms of a general form. Results are applied to time decay estimates for the Fokker–Planck equation and for semilinear hyperbolic equations.

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# 1 Introduction

These notes are devoted to the investigation of dispersive and Strichartz estimates for general hyperbolic equations with constant coefficients. The analysis that we carry out is also applicable to hyperbolic systems either by looking at characteristics

of the system directly, or first taking the determinant of the system (the dispersion relation).

There are several important motivations for the analysis. First, while hyperbolic equations of the second order (such as the wave equation, dissipative wave equation, Klein–Gordon equation, etc.) are very well studied, relatively little is known about equations of higher orders. At the same time, equations or systems of high orders naturally arise in applications. For example, Grad systems of non-equilibrium gas dynamics, when linearised near an equilibrium point, are examples of large hyperbolic systems with constant coefficients (see e.g. [Rad03], [Rad05]). Here one has to deal with hyperbolic equations of orders 13, 20, etc., depending on the number of moments in the Grad system. Moreover, there are important families of systems of size going to infinity, or even of infinite hyperbolic systems. For example, the Hermite–Grad method for the analysis of the Fokker–Planck equation for the distribution function for particles for the Brownian motion produces an infinite hyperbolic system with constant coefficients. Indeed, making the decomposition in the space of velocities into the Hermite basis, and writing equations for the space-time coefficients produces a hyperbolic system for infinitely many coefficients (see e.g. [VR03], [VR04], [ZR04], and Section 8.5). The Galerkin approximation of this system leads to a family of systems with sizes increasing to infinity. Although explicit calculations are difficult in these situations, the time decay rate of the solution can still be calculated ([Ruzh06]).

One of the main difficulties when dealing with large systems is that unlike in the case of the second order equations, in general characteristics can not be calculated explicitly. This raises a natural problem to look for properties of the equation that determine the decay rates for solutions. On one hand, it becomes clear that one has to look for geometric properties of characteristics that may be responsible for such decay rates. On the other hand, a subsequent problem arises to be able to reduce these properties from some properties of coefficients of the equation.

One encounters several difficulties on this path. One difficulty lies in the absence of general formulae for characteristic roots. For large frequencies one can use perturbation methods to deduce the necessary asymptotic properties of characteristics. However, this approach can not be used for small frequencies, where the situation becomes more subtle. For example, for small frequencies characteristics may become multiple, causing them to become irregular. This means that if we use the usual representation of solutions in terms of Fourier multipliers, phases become irregular, while amplitudes are irregular and blow up. Thus, we will need to carry out the detailed analysis of sets of possible multiplicities using the fact that they are solutions of parameter dependent polynomial equations. Another difficulty for small frequencies is that there exists a genuine interaction between time and frequencies. In the case of homogeneous symbols it can be shown (see e.g. Section 1.2) that time can be taken out of the estimates, after which low frequencies can be ignored since the corresponding operators are smoothing and their estimates are independent of time. In the case of the presence of lower order terms, the time can no longer be eliminated from the estimates, so even small frequencies become large for large times and may influence the resulting estimates.

The purpose of this work is to present a comprehensive analysis of such problems. Despite the difficulties described above, we will be able to determine what geometric properties of characteristic roots are responsible for qualitatively different time decay rates for solutions. Moreover, we will calculate these rates and relate them to geometric properties of equations. This will lead to a comprehensive picture of decay rates and orders in dispersive estimates for hyperbolic equations with constant coefficients. Such estimates lead to Strichartz estimates, for which our analysis will be applied, with further implications for the corresponding semilinear problems.

Thus, in this paper we consider a problem of determining dispersive and Strichartz estimates for general hyperbolic equations with lower order terms. Therefore, we consider the Cauchy problem for general  $m^{\text{th}}$  order constant coefficient linear strictly hyperbolic equation with solution  $u = u(t, x)$ :

$$\begin{cases} \overbrace{D_t^m u + \sum_{j=1}^m P_j(D_x) D_t^{m-j} u}^{\text{homogeneous principal part}} + \overbrace{\sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u}^{\text{general lower order terms}} = 0, & t > 0, \\ D_t^l u(0, x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), & l = 0, \dots, m-1, x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where  $P_j(\xi)$ , the polynomial obtained from the operator  $P_j(D_x)$  by replacing each  $D_{x_k}$  by  $\xi_k$ , is a constant coefficient homogeneous polynomial of order  $j$ , and the  $c_{\alpha,r}$  are (complex) constants. Here, as usual,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$ ,  $D_{x_k} = \frac{1}{i} \partial_{x_k}$  and  $D_t = \frac{1}{i} \partial_t$ . The full symbol of the operator in (1.1) will be denoted by

$$L(\tau, \xi) = \tau^m + \sum_{j=1}^m P_j(\xi) \tau^{m-j} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha \tau^r,$$

where  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ . We will always assume that the differential operator in (1.1) is hyperbolic, that is for each  $\xi \in \mathbb{R}^n$ , the symbol of the principal part,

$$L_m(\tau, \xi) = \tau^m + \sum_{j=1}^m P_j(\xi) \tau^{m-j},$$

has  $m$  real roots with respect to  $\tau$ . For simplicity, unless explicitly stated otherwise, we will also assume that the operator in (1.1) is *strictly hyperbolic*, that is at each  $\xi \in \mathbb{R}^n \setminus \{0\}$ , these roots are pairwise distinct. We denote the roots of  $L_m(\tau, \xi)$  with respect to  $\tau$  by  $\varphi_1(\xi) \leq \dots \leq \varphi_m(\xi)$ , and if  $L$  is strictly hyperbolic the above inequalities are strict for  $\xi \neq 0$ .

The condition of hyperbolicity arises naturally in the study of the Cauchy problem for linear partial differential operators and it can be shown that it is a necessary condition for  $C^\infty$  well-posedness of the problem; this is discussed in [ES92] and [Hör83b], for example. Strict hyperbolicity is sufficient for  $C^\infty$  well-posedness of the Cauchy problem for such an operator with any lower order terms; if the operator is only hyperbolic (sometimes called *weakly hyperbolic*) the lower order terms must satisfy additional conditions for  $C^\infty$  well-posedness, the so-called *Levi conditions*. For this

reason, we only consider strictly hyperbolic operators with lower order terms, since our main interest is to understand the influence of lower order terms on the decay properties of solutions.

The roots of the associated full characteristic polynomial  $L(\tau, \xi)$  with respect to  $\tau$  will be denoted by  $\tau_1(\xi), \dots, \tau_m(\xi)$  and referred to as the *characteristic roots* of the full operator. Clearly, if  $L$  is a homogeneous operator then the characteristic roots  $\tau_k(\xi)$ ,  $k = 1, \dots, m$ , coincide, possibly after reordering, with the roots  $\varphi_k(\xi)$ ,  $k = 1, \dots, m$ , of the operator  $L_m$ . However, in general there is no natural ordering on the roots  $\tau_k(\xi)$  as they may be complex-valued or may intersect.

The analysis here will be based on the properties of characteristic roots  $\tau_k(\xi)$ . If the problem (1.1) is strictly hyperbolic, we can derive their asymptotic properties in a general situation, necessary for our analysis. However, if the problem is only hyperbolic, functions  $\tau_k(\xi)$  may develop singularities for large  $\xi$ . If this does not happen and we have the necessary information about them, we may drop the strict hyperbolicity assumption. This may be the case in some applications, for example in those arising in the analysis of the Fokker–Planck equation.

We seek *a priori* estimates for the solution  $u(t, x)$  to the Cauchy problem (1.1), of the type

$$\|D_x^\alpha D_t^r u(t, \cdot)\|_{L^q} \leq K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}}, \quad (1.2)$$

where  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $N_p = N_p(\alpha, r)$  is a constant depending on  $p, \alpha$  and  $r$ , and  $K(t)$  is a function to be determined. Here  $W_p^{N_p-l}$  is the Sobolev space over  $L^p$  with  $N_p - l$  (fractional) derivatives.

We note that sometimes, for example in [Trè80], in the definition of a hyperbolic operator the polynomial  $L(i\tau, \xi)$  is used as it is better suited to taking the partial Fourier transform in  $x$ , corresponding as it does to  $L(\partial_t, D_x)$ ; in this case, one requires the roots with respect to  $\tau$  to be purely imaginary (in the cases when we will require them to be real). However, the definition that we give above is perhaps more standard, and thus adopted here throughout.

For a hyperbolic equation with real coefficients we note that the constants  $c_{\alpha,r}$  satisfy  $i^{m-|\alpha|-l} c_{\alpha,r} \in \mathbb{R}$ ; the equation is written in the form above since our results may be used to study hyperbolic systems, which can be reduced to an  $m^{\text{th}}$  order equation with complex coefficients.

Most results presented here will apply to operators which are pseudo-differential in  $x$  and to hyperbolic systems via their dispersion equation. Moreover, most of results in this paper are in general sharp.

In this work, we place the priority on obtaining a comprehensive collection of estimates for hyperbolic equations with constant coefficients. The case of variable coefficients is also of great interest, but we leave some extensions of our analysis to this case outside the scope of this paper. Let us mention that already in the case of coefficients depending on time, some unpleasant phenomena may happen. For example, already for the second order equations the oscillations in time dependent coefficients may change the time decay rates for solutions to the corresponding Cauchy problem.

For example, equations with very fast oscillations, or with increasing coefficients, have been analysed in [RY99, RY00], to mention only a few references. Results even for the wave equations with bounded coefficients may depend on the oscillations in coefficients (see e.g. [ReS05]). At the same time, many results of this paper are stable under time perturbations of coefficients. For example, in the case of equations with homogeneous symbols with time-dependent coefficients with integrable derivative, a comprehensive analysis has been carried out in [MR07]. We will not deal with such questions in this paper. Let us also mention that while dispersive estimates are devoted to  $L^p - L^q$  estimates for solutions,  $L^p - L^p$  estimates are also of interest. A survey of  $L^p$  estimates for general non-degenerate Fourier integral operators and their dependence on the geometry can be found in [Ruzh00] in the case of real-valued phase functions, while operators with complex-valued phase functions have been analysed in [Ruzh01].  $L^p$ -estimates for solutions to some classes of hyperbolic systems with variable multiplicities appeared in [KR07].

Let us now explain the organisation of these notes. In the following parts of the introduction we will review results for second order equations and for equations with homogeneous symbols, as well as give several more motivations for the comprehensive analysis of this paper. In Section 2 we will present results for different types of behaviour of characteristic roots, and also of corresponding phase functions in cases where we can represent solutions in terms of Fourier multipliers. Thus, in Section 2.1 we will present results without and with multiplicities, when roots are separated from the real axis, in which cases we can get exponential decay of solutions. In Section 2.2 we present results for roots with non-degeneracies, in which case we have a variety of conclusions depending on geometric properties of roots. In Section 2.3 we present results for complex roots that become real on some set. A version of this type of statements (although not in the microlocal form used here) partly appeared in [RS05], and those are improved here. In Section 2.4 we summarise the microlocal results and formulate the main theorem on dispersive estimates for general hyperbolic equations with constant coefficients. Theorem 2.18 is the main theorem containing a table of results, and the rest of this section is devoted to the explanation and further remarks about this table. In Section 2.5 we will outline our approach, indicating the relations between frequency regions and statements. In Section 2.6 we present results for non-homogeneous equations, as well as formulate corresponding Strichartz estimates with further applications to semilinear equations. In general, we leave such developments outside the scope of this paper since they are quite well understood (see e.g. [KT98]), once the time decay rates are determined (as we will do in Theorem 2.18).

The subsequent chapters contain the detailed analysis and proofs. In Section 3 we establish necessary properties of roots of hyperbolic polynomials, as well as carry out the perturbation analysis for large frequencies. In Section 4 we investigate estimates for oscillatory integrals under certain convexity assumptions on the level sets of the phase function. In Section 5 we analyse the corresponding oscillatory integrals without convexity assumption. Section 6 is devoted to dispersive estimates for solutions to the general Cauchy problem, and here we prove various parts of Theorem 2.18.



Section 7 deals with multiple characteristics. Here we present a procedure for the resolution of multiplicities in the representation of solutions, enabling us to obtain estimates in these cases as well. Section 7.4 is devoted to multiple roots on the real axis. Here, we investigate solutions for frequencies very close to multiplicities (in some shrinking neighborhoods) as well as for larger, but still bounded, frequencies. Here we present several different versions of results dependent on possibly different assumptions. Finally, Section 8 is devoted to examples of the presented analysis with further applications. Thus, in Section 8.1 we deal with second order equations and give examples of how our results can be applied to investigate the interplay between mass, dissipation, and frequencies. Further, in Section 8.2 we discuss some conditions on coefficients of equations, and in Section 8.3 we give examples of non-homogeneous roots in terms of hyperbolic triples and Hermite's theorem. In Section 8.4 we show briefly how the results can be applied for strictly hyperbolic systems. And finally, in Section 8.5 we give an application to the Fokker–Planck equations.

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We will denote various constants throughout the paper by the same letter  $C$ . Balls with radius  $R$  centred at  $\xi \in \mathbb{R}^n$  will be denoted by  $B_R(\xi)$ . We will use the notation  $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ ,  $\langle D \rangle = \sqrt{1 - \Delta}$  and  $|D| = |-\Delta|^{1/2}$ . The Sobolev space  $W_p^l$  is then defined as the space of measurable functions for which  $\langle D \rangle^l f \in L^p(\mathbb{R}_x^n)$ .

We will also use the standard notation for the symbol class  $S^\mu = S_{1,0}^\mu$ , as a space of smooth functions  $a = a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  satisfying symbolic estimates  $|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{\mu - |\alpha|}$ , for all  $x, \xi \in \mathbb{R}^n$ , and all multi-indices  $\alpha, \beta$ .

If function  $a = a(\xi)$  is independent of  $x$ , we will sometimes also write  $a \in S_{1,0}^\mu(U)$  for an open set  $U \subset \mathbb{R}^n$ , if  $a = a(\xi) \in C^\infty(U)$  satisfies  $|\partial_\xi^\alpha a(\xi)| \leq C_\alpha(1 + |\xi|)^{\mu - |\alpha|}$ , for all  $\xi \in U$ , and all multi-indices  $\alpha$ .

## 1.1 Background

The study of  $L^p - L^q$  decay estimates, or Strichartz estimates, for linear evolution equations began in 1970 when Robert Strichartz published two papers, [Str70a] and [Str70b]. He proved that if  $u = u(t, x)$  satisfies the Cauchy problem (that is, the initial value problem) for the homogeneous linear wave equation

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ u(0, x) = \phi(x), \quad \partial_t u(0, x) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where the initial data  $\phi$  and  $\psi$  lie in suitable function spaces such as  $C_0^\infty(\mathbb{R}^n)$ , then the *a priori* estimate

$$\|(u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^q} \leq C(1 + t)^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}} \quad (1.4)$$

holds when  $n \geq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$  and  $N_p \geq n(\frac{1}{p} - \frac{1}{q})$ . Using this estimate, Strichartz proved global existence and uniqueness of solutions to the Cauchy problem

for nonlinear wave equations with suitable (“small”) initial data. This procedure of proving an *a priori* estimate for a linear equation and using it, together with local existence of a nonlinear equation, to prove global existence and uniqueness for a variety of nonlinear evolution equations is now standard; a systematic overview, with examples including the equations of elasticity, Schrödinger equations and heat equations, can be found in [Rac92], or in many other more recent books.

There are two main approaches used in order to prove (1.4); firstly, one may write the solution to (1.3) using the d’Alembert ( $n = 1$ ), Poisson ( $n = 2$ ) or Kirchhoff ( $n = 3$ ) formulae, and their generalisation to large  $n$ ,

$$u(t, x) = \begin{cases} \frac{1}{\prod_{j=1}^{\frac{n-1}{2}} (2j-1)} \left[ \partial_t(t^{-1}\partial_t)^{\frac{n-3}{2}} \left( t^{n-1} \oint_{\partial B_t(x)} \phi dS \right) \right. \\ \quad \left. + (t^{-1}\partial_t)^{\frac{n-3}{2}} \left( t^{n-1} \oint_{\partial B_t(x)} \psi dS \right) \right] & (\text{odd } n \geq 3) \\ \frac{1}{\prod_{j=1}^{n/2} 2j} \left[ \partial_t(t^{-1}\partial_t)^{\frac{n-2}{2}} \left( t^n \int_{B_t(x)} \frac{\phi(y)}{\sqrt{t^2 - |y-x|^2}} dy \right) \right. \\ \quad \left. + (t^{-1}\partial_t)^{\frac{n-2}{2}} \left( t^n \int_{B_t(x)} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} dy \right) \right] & (\text{even } n), \end{cases}$$

(here  $\oint$  stands for the averaged integral; for the derivation of these formulae see, for example, [Ev98]), as is done in [vW71] and [Rac92]. Alternatively, one may write the solution as a sum of Fourier integral operators:

$$u(t, x) = \mathcal{F}^{-1} \left( \frac{e^{it|\xi|} + e^{-it|\xi|}}{2} \widehat{\phi}(\xi) + \frac{e^{it|\xi|} - e^{-it|\xi|}}{2|\xi|} \widehat{\psi}(\xi) \right).$$

This is done in [Str70a], [Bre75] and [Pec76], for example. Using one of these representations for the solution and techniques from either the theory of Fourier integral operators ([Pec76]), Bessel functions ([Str70a]), or standard analysis ([vW71]), the estimate (1.4) may be obtained.

Let us now compare the time decay rate for the wave equation with equations with lower order terms. An important example is the Klein–Gordon equation, where  $u = u(t, x)$  satisfies the initial value problem

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) + \mu^2 u(t, x) = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.5)$$

where  $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ , say, and  $\mu \neq 0$  is a constant (representing a *mass term*); then

$$\|(u(t, \cdot), u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}}, \quad (1.6)$$

where  $p, q, N_p$  are as before. Comparing (1.4) to (1.6), we see that the estimate for the solution to the Klein–Gordon equation decays more rapidly. The estimate is



proved in [vW71], [Pec76] and [Hör97] in different ways, each suggesting reasons for this improvement: in [vW71], the function

$$v = v(x, x_{n+1}, t) := e^{-i\mu x_{n+1}} u(t, x), \quad x_{n+1} \in \mathbb{R},$$

is defined; using (1.5), it is simple to show that  $v$  satisfies the wave equation in  $\mathbb{R}^{n+1}$ , and thus the Strichartz estimate (1.4) holds for  $v$ , yielding the desired estimate for  $u$ . This is elegant, but cannot easily be adapted to other situations due to the importance of the structures of the Klein–Gordon and wave equations for this proof. In [Pec76] and [Hör97], a representation of the solution via Fourier integral operators is used and the stationary phase method then applied in order to obtain estimate (1.6).

Another second order problem of interest is the Cauchy problem for the dissipative wave equation,

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) + u_t(t, x) = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.7)$$

where  $\psi, \phi \in C_0^\infty(\mathbb{R}^n)$ , say. In this case,

$$\|\partial_t^r \partial_x^\alpha u(t, \cdot)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-r-\frac{|\alpha|}{2}} \|(\phi, \nabla \psi)\|_{W_p^{N_p}}, \quad (1.8)$$

with some  $N_p = N_p(n, \alpha, r)$ . This is proved in [Mat77] with a view to showing well-posedness of related semilinear equations. Once again, this estimate (for the solution  $u(t, x)$  itself) is better than that for the solution to the wave equation; there is an even greater improvement for higher derivatives of the solution. As before, the proof of this may be done via a representation of the solution using the Fourier transform:

$$u(t, x) = \begin{cases} \mathcal{F}^{-1} \left( \left[ \frac{e^{-t/2} \sinh \left( \frac{t}{2} \sqrt{1-4|\xi|^2} \right)}{\sqrt{1-4|\xi|^2}} + e^{-t/2} \cosh \left( \frac{t}{2} \sqrt{1-4|\xi|^2} \right) \right] \widehat{\phi}(\xi) \right. \\ \quad \left. + \frac{2e^{-t/2} \sinh \left( \frac{t}{2} \sqrt{1-4|\xi|^2} \right)}{\sqrt{1-4|\xi|^2}} \widehat{\psi}(\xi) \right), & |\xi| \leq 1/2, \\ \mathcal{F}^{-1} \left( \left[ \frac{e^{-t/2} \sin \left( \frac{t}{2} \sqrt{4|\xi|^2-1} \right)}{\sqrt{4|\xi|^2-1}} + e^{-t/2} \cos \left( \frac{t}{2} \sqrt{4|\xi|^2-1} \right) \right] \widehat{\phi}(\xi) \right. \\ \quad \left. + \frac{2e^{-t/2} \sin \left( \frac{t}{2} \sqrt{4|\xi|^2-1} \right)}{\sqrt{4|\xi|^2-1}} \widehat{\psi}(\xi) \right), & |\xi| > 1/2. \end{cases}$$

Matsumura divides the phase space into the regions where the solution has different properties and then uses standard techniques from analysis.

It is, therefore, motivating to ask why the addition of lower order terms improves the rate of decay of the solution to the equation; furthermore, in the first instance, we would like to understand why the improvement in the decay is the same for both the addition of a mass term and for the addition of a dissipative term. It will follow from the analysis of the paper that the quantities responsible for the decay rates for the Klein-Gordon and dissipative equations are of completely different nature. In the first

instance the characteristic roots are real and lie on the real axis for all frequencies, while for the latter equation they are in the upper complex half-plane, intersect at a point, and one of them comes to the origin. From this point of view, the same decay rates in the dispersive estimate for these two equations is quite a coincidence. On the example of the dissipative equation we can see another difficulty for the analysis, namely the appearance of the multiple roots. This may lead to the loss of regularity in roots and blow-ups in the amplitudes of a representation, so we need to develop some techniques to deal with this type of situations.

These questions are even more important for equations of higher orders. Let us mention briefly an example of a system that arises as the linearisation of the 13-moment Grad system of non-equilibrium gas dynamics in two dimensions (other Grad systems are similar). The dispersion relation (the determinant) of this system is a polynomial of 9<sup>th</sup> order that can be written as

$$P = Q_9 - iQ_8 - Q_7 + iQ_6 + Q_5 - iQ_4,$$

with polynomials  $Q_j(\omega, \xi)$  defined by

$$\begin{aligned} Q_9(\omega, \xi) &= |\xi|^9 \omega^3 \left[ \omega^6 - \frac{103}{25} \omega^4 + \frac{21}{5} \omega^2 \left( 1 - \frac{912}{2625} \alpha \beta \right) - \frac{27}{25} \left( 1 - \frac{432}{675} \alpha \beta \right) \right], \\ Q_8(\omega, \xi) &= |\xi|^8 \omega^2 \left[ \frac{13}{3} \omega^6 - \frac{1094}{75} \omega^4 + \frac{1381}{125} \omega^2 \left( 1 - \frac{2032}{6905} \alpha \beta \right) - \frac{264}{125} \left( 1 - \frac{143}{330} \alpha \beta \right) \right], \\ Q_7(\omega, \xi) &= |\xi|^7 \omega \left[ \frac{67}{9} \omega^6 - \frac{497}{25} \omega^4 + \frac{3943}{375} \omega^2 \left( 1 - \frac{832}{3943} \alpha \beta \right) - \frac{159}{125} \left( 1 - \frac{48}{159} \alpha \beta \right) \right], \\ Q_6(\omega, \xi) &= |\xi|^6 \left[ \frac{19}{3} \omega^6 - \frac{2908}{225} \omega^4 + \frac{13}{3} \omega^2 \left( 1 - \frac{32}{325} \alpha \beta \right) - \frac{6}{25} \right], \\ Q_5(\omega, \xi) &= |\xi|^5 \omega \left[ \frac{8}{3} \omega^4 - \frac{178}{45} \omega^2 + \frac{2}{3} \right], \\ Q_4(\omega, \xi) &= \frac{4}{9} |\xi|^4 \omega^2 (\omega^2 - 1), \end{aligned}$$

where

$$\omega(\xi) = \frac{\tau(\xi)}{|\xi|}, \quad \alpha = \frac{\xi_1^2}{|\xi|^2}, \quad \beta = \frac{\xi_2^2}{|\xi|^2}.$$

A natural question of finding dispersive (and subsequent Strichartz) estimates for the Cauchy problem for operator  $P(D_t, D_x)$  with symbol  $P(\tau, \xi)$  becomes computationally complicated. Clearly, in this situation it is hard to find the roots explicitly, and, therefore, we need some procedure of determining what are the general properties of the characteristics roots, and how to derive the time decay rate from these properties. Thus, in [Rad03] and [VR04] it is discussed when such polynomials are stable. In this case, the analysis of this paper will guarantee the decay rate, e.g. by applying Theorem 2.16 for frequencies near the origin, Theorem 2.2 for bounded frequencies near possible multiplicities (independent of the structure of such multiplicities), and Theorem 2.1 for large frequencies. In fact, once the behavior of the characteristic roots is understood, Theorem 2.18 will immediately show that the overall time decay rate here is the same as for the dissipative wave equation.

## 1.2 Homogeneous symbols

The case where the operator in (1.1) has homogeneous symbol has been studied extensively:

$$\begin{cases} L_m(D_x, D_t)u = 0, & (t, x) \in \mathbb{R}^n \times (0, \infty), \\ D_t^l u(0, x) = f_l(x), & l = 0, \dots, m-1, x \in \mathbb{R}^n, \end{cases} \quad (1.9)$$

where  $L_m$  is a homogeneous  $m^{\text{th}}$  order constant coefficient strictly hyperbolic differential operator; the symbol of  $L_m$  may be written in the form

$$L_m(\tau, \xi) = (\tau - \varphi_1(\xi)) \dots (\tau - \varphi_m(\xi)), \text{ with } \varphi_1(\xi) < \dots < \varphi_m(\xi) \quad (\xi \neq 0).$$

In a series of papers, [Sug94], [Sug96] and [Sug98], Sugimoto showed how the geometric properties of the characteristic roots  $\varphi_1(\xi), \dots, \varphi_m(\xi)$  affect the  $L^p - L^q$  estimate. To understand this, let us summarise the method of approach.

Firstly, the solution can be written as the sum of Fourier multipliers:

$$u(t, x) = \sum_{l=0}^{m-1} [E_l(t)f_l](x), \quad \text{where } E_l(t) = \sum_{k=1}^m \mathcal{F}^{-1} e^{it\varphi_k(\xi)} a_{k,l}(\xi) \mathcal{F},$$

and  $a_{k,l}(\xi)$  is homogeneous of order  $-l$ . Now, the problem of finding an  $L^p - L^q$  decay estimate for the solution is reduced to showing that operators of the form

$$M_r(D) := \mathcal{F}^{-1} e^{i\varphi(\xi)} |\xi|^{-r} \chi(\xi) \mathcal{F},$$

where  $\varphi(\xi) \in C^\omega(\mathbb{R}^n \setminus \{0\})$  is homogeneous of order 1 and  $\chi \in C^\infty(\mathbb{R}^n)$  is equal to 1 for large  $\xi$  and zero near the origin, are  $L^p - L^q$  bounded for suitably large  $r \geq l$ . In particular, this means that, for such  $r$ , we have

$$\|E_l(1)f\|_{L^q} \leq C\|f\|_{W_p^{r-l}}.$$

Then it may be assumed, without loss of generality, that  $t = 1$ . Indeed, it can be readily checked that for  $t > 0$  and  $f \in C_0^\infty(\mathbb{R}^n)$ , we have the equality

$$[E_l(t)f](x) = t^l [E_l(1)f(t \cdot)](t^{-1}x).$$

Using this identity and denoting  $f_t(\cdot) = f(t \cdot)$ , we have

$$\begin{aligned} \|E_l(t)f\|_{L^q}^q &= t^{lq} \|[E_l(1)f_t](t^{-1} \cdot)\|_{L^q}^q = t^{lq} \int_{\mathbb{R}^n} |[E_l(1)f_t](t^{-1}x)|^q dx \\ &\stackrel{(x=tx')}{=} t^{lq} \int_{\mathbb{R}^n} t^n |[E_l(1)f_t](x')|^q dx' = t^{lq+n} \|E_l(1)f_t\|_{L^q}^q. \end{aligned}$$

Then, noting that a simple change of variables yields

$$\|f_t\|_{W_p^k}^p \leq C t^{kp-n} \|f\|_{W_p^k}^p,$$

we have,

$$\|E_l(t)f\|_{L^q} \leq C t^{l+\frac{n}{q}} \|f_t\|_{W_p^{r-l}} \leq C t^{r-n(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{r-l}};$$

hence,

$$\|u(t, \cdot)\|_{L^q} \leq C t^{r-n(\frac{1}{p}-\frac{1}{q})} \sum_{l=0}^{m-1} \|f_l\|_{W_p^{r-l}}.$$

It has long been known that the values of  $r$  for which  $M_r(D)$  is  $L^p - L^q$  bounded depend on the geometry of the level set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\}.$$

In [Lit73], [Bre75], it is shown that if the Gaussian curvature of  $\Sigma_\varphi$  is never zero then  $M_r(D)$  is  $L^p - L^q$  bounded when  $r \geq \frac{n+1}{2}(\frac{1}{p} - \frac{1}{q})$ . This is extended in [Bre77] where it is proven that  $M_r(D)$  is  $L^p - L^q$  bounded provided  $r \geq \frac{2n-\rho}{2}(\frac{1}{p} - \frac{1}{q})$ , where  $\rho = \min_{\xi \neq 0} \text{rank Hess } \varphi(\xi)$ .

Sugimoto extended this further in [Sug94], where he showed that if  $\Sigma_\varphi$  is convex then  $M_r(D)$  is  $L^p - L^q$  bounded when  $r \geq (n - \frac{n-1}{\gamma(\Sigma_\varphi)})(\frac{1}{p} - \frac{1}{q})$ ; here,

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_P \gamma(\Sigma; \sigma, P), \quad \Sigma \subset \mathbb{R}^n \text{ a hypersurface},$$

where  $P$  is a plane containing the normal to  $\Sigma$  at  $\sigma$  and  $\gamma(\Sigma; \sigma, P)$  denotes the order of the contact between the line  $T_\sigma \cap P$ ,  $T_\sigma$  is the tangent plane at  $\sigma$ , and the curve  $\Sigma \cap P$ . See Section 4.3 for more on this maximal order of contact.

In order to apply this result to the solution of (1.9), it is necessary to find a condition under which the level sets of the characteristic roots are convex. The following notion is the one that is sufficient:

**Definition 1.1.** *Let  $L = L(D_t, D_x)$  be a homogeneous  $m^{\text{th}}$  order constant coefficient partial differential operator. It is said to satisfy the convexity condition if the matrix of the second order derivatives,  $\text{Hess } \varphi_k(\xi)$ , corresponding to each of its characteristic roots  $\varphi_1(\xi), \dots, \varphi_m(\xi)$ , is semi-definite for  $\xi \neq 0$ .*

It can be shown that if an operator  $L$  does satisfy this convexity condition, then the above results can be applied to the solution and thus an estimate of the form (1.2) holds with

$$K(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}, \quad \text{with some } \gamma \leq m, \quad (1.10)$$

where  $\gamma$  can be related to the convex indices of the level sets of characteristics. Indeed, under the convexity condition one can show that  $\phi_k$  can be made always positive or negative by adding an affine function, the corresponding level sets  $\Sigma_{\phi_k} = \{\xi \in \mathbb{R}^n : \phi_k(\xi) = 1\}$  are convex for each  $k = 1, \dots, m$ , and that  $\gamma(\Sigma_{\phi_k}) \leq 2[m/2]$ . So the decay in (1.10) is guaranteed with  $\gamma = 2[m/2]$ .

Finally, if this convexity condition does not hold the estimate fails; in [Sug96] and [Sug98] it is shown that in general,  $M_r(D)$  is  $L^p - L^q$  bounded when  $r \geq (n - \frac{1}{\gamma_0(\Sigma_\varphi)})(\frac{1}{p} - \frac{1}{q})$ , where

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_P \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma).$$

For  $n = 2$ ,  $\gamma_0(\Sigma) = \gamma(\Sigma)$ , so, the convexity condition may be lifted in that case. However, in [Sug96], examples are given when  $n \geq 3$ ,  $p = 1, 2$  where this lower bound for  $r$  is the best possible and, thus, the convexity condition is necessary for the above estimate. It turns out that the case  $n \geq 3$ ,  $1 < p < 2$  is more interesting and is studied in greater depth in [Sug98], where microlocal geometric properties must be looked at in order to obtain an optimal result.

Two remarks are worth making; firstly, the convexity condition result recovers the Strichartz decay estimate for the wave equation, since that clearly satisfies such a condition. Secondly, the convexity condition is an important restriction on the geometry of the characteristic roots that affects the  $L^p - L^q$  decay rate; hence, in the case of an  $m^{\text{th}}$  order operator with lower order terms we must expect some geometrical conditions on the characteristic roots to affect the decay rate of solutions.

## 2 Main results

We will now turn to analysing the conditions under which we can obtain  $L^p - L^q$  decay estimates for the general  $m^{\text{th}}$  order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$\begin{cases} L(D_t, D_x) \equiv D_t^m u + \sum_{j=1}^m P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, & t > 0, \\ D_t^l u(0, x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), & l = 0, \dots, m-1, x \in \mathbb{R}^n. \end{cases} \quad (2.1)$$

Results of this section will show how different behaviours of the characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  affect the rate of decay that can be obtained. As in the introduction, the symbol  $P_j(\xi)$  of  $P_j(D_x)$  is a homogeneous polynomial of order  $j$ , and the  $c_{\alpha,r}$  are constants. The differential operator in the first line of (2.1) will be denoted by  $L(D_t, D_x)$  and its symbol by  $L(\tau, \xi)$ . The principal part of  $L$  is denoted by  $L_m$ . Thus,  $L_m(\tau, \xi)$  is a homogeneous polynomial of order  $m$ . In the subsequent analysis, ideally, of course, we would like to have conditions on the lower order terms for different rates of decay; in Section 8 we shall give some results in this direction. For now, though, we concentrate on conditions on the characteristic roots.

First of all, it is natural to impose the stability condition, namely that for all  $\xi \in \mathbb{R}^n$  we have

$$\text{Im } \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m; \quad (2.2)$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points  $\xi \in \mathbb{R}^n$ , and thus cannot be expected to be lifted. In fact, certain microlocal decay estimates are possible even without this condition if the supports of the Fourier transforms of the Cauchy data are contained in the set where condition (2.2) holds. However, this restriction is only technical so we may assume (2.2) without great loss of generality since otherwise no time decay of solution can be expected.

Also, it is sensible to divide the considerations of how characteristic roots behave into two parts: their behaviour for large values of  $|\xi|$  and for bounded values of  $|\xi|$ .

These two cases are then subdivided further; in particular the following are the key properties to consider:

- multiplicities of roots (this only occurs in the case of bounded frequencies  $|\xi|$ );
- whether roots lie on the real axis or are separated from it;
- behaviour as  $|\xi| \rightarrow \infty$  (only in the case of large  $|\xi|$ );
- how roots meet the real axis (if they do);
- properties of the Hessian of the root,  $\text{Hess } \tau_k(\xi)$ ;
- a convexity-type condition, as in the case of homogeneous roots (Section 1.2).

For some frequencies away from multiplicities we can actually establish independently interesting estimates for the corresponding oscillatory integrals that contribute to the solution. Around multiplicities we need to take extra care of the structure of solutions. This will be done by dividing the frequencies into zones each of which will give a certain decay rate. Combined together they will yield the total decay rate for solution to (2.1). Several theorems below will deal with integrals of the form

$$\int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \chi(\xi) d\xi, \quad (2.3)$$

which appear in representations of solutions to Cauchy problem (2.1) as kernels of propagators, where  $a(\xi)$  is a suitable amplitude and  $\chi(\xi)$  is a cut-off to a corresponding zone, which may be bounded or unbounded. Solution to the Cauchy problem (2.1) can be written in the form

$$u(t, x) = \sum_{j=0}^{m-1} E_j(t) f_j(x),$$

where propagators  $E_j(t)$  are defined by

$$E_j(t) f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi, \quad (2.4)$$

with suitable amplitudes  $A_j^k(t, \xi)$ . In the areas where roots are simple, phases and amplitudes are smooth, and we can analyse the sum (2.4) termwise, reducing the analysis to integrals of the form (2.3). In the case of multiple characteristics we will group terms in (2.4) in a special way to obtain suitable decay estimates. Below we will give results for decay rates dependent on the different qualitative behaviours of the characteristic roots.

## 2.1 Away from the real axis: exponential decay

We begin by looking at the zone where roots are separated from the real axis. If the roots are smooth, we can analyse solution (2.4) termwise:

**Theorem 2.1.** *Let  $\tau : U \rightarrow \mathbb{C}$  be a smooth function,  $U \subset \mathbb{R}^n$  open. Let  $a \in S_{1,0}^{-\mu}(U)$ , i.e. assume that  $a = a(\xi) \in C^\infty(U)$  satisfies  $|\partial_\xi^\alpha a(\xi)| \leq C_\alpha(1 + |\xi|)^{-\mu-|\alpha|}$ , for all  $\xi \in U$  and all multi-indices  $\alpha$ . Let  $\chi \in S_{1,0}^0(\mathbb{R}^n)$  be such that  $\chi = 0$  outside  $U$ . Assume further that:*

- (i) *there exists  $\delta > 0$  such that  $\text{Im } \tau(\xi) \geq \delta$  for all  $\xi \in U$ ;*
- (ii)  *$|\tau(\xi)| \leq C(1 + |\xi|)$  for all  $\xi \in U$ .*

Then for all  $t \geq 0$  we have

$$\left\| D_t^r D_x^\alpha \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^q(\mathbb{R}_x^n)} \leq C e^{-\delta t} \|f\|_{W_p^{N_p + |\alpha| + r - \mu}}, \quad (2.5)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$ ,  $N_p \geq n(\frac{1}{p} - \frac{1}{q})$ ,  $r \geq 0$ ,  $\alpha$  a multi-index and  $f \in C_0^\infty(\mathbb{R}^n)$ . If  $p = 1$ , we take  $N_1 > n$ .

Moreover, let us assume that equation  $L(\tau, \xi) = 0$  has only simple roots  $\tau_k(\xi)$  which satisfy condition (i) above, in the open set  $U \subset \mathbb{R}^n$ , for all  $k = 1, \dots, m$ . Then solution  $u$  to (2.1) satisfies

$$\|D_t^r D_x^\alpha \chi(D) u(t, \cdot)\|_{L^q(\mathbb{R}_x^n)} \leq C e^{-\delta t} \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p + |\alpha| + r - l}}, \quad (2.6)$$

where  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $N_p, r, \alpha$  are as above.

The proof of Theorem 2.1 will be given in Sections 6.4 and 6.10. Note also that if we omit assumption (ii) in Theorem 2.1, estimate (2.5) with  $r = 0$  still holds. In the case of (2.6), it can be shown (see Proposition 3.8) that characteristic roots of operator  $L(D_t, D_x)$  in (2.1) satisfy (ii).

We also note, that we may have different norms on the right hand side of (2.6). For example, we will show in Section 6.4, that under conditions of Theorem 2.1 we also have the following estimate:

$$\|D_t^r D_x^\alpha \chi(D) u(t, \cdot)\|_{L^q(\mathbb{R}_x^n)} \leq C e^{-\delta t} \sum_{l=0}^{m-1} \|f_l\|_{W_2^{N'_q + |\alpha| + r - l}}, \quad (2.7)$$

where  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $N'_q \geq \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ , and  $N'_\infty > \frac{n}{2}$  for  $p = 1$ . Estimate (2.7) will follow from (6.8) and Proposition 6.5 by interpolation. In turn, interpolating between (2.6) and (2.7), we can obtain similar  $L^p - L^q$  estimates for all intermediate  $p$  and  $q$ .

To be able to derive time decay in the case of multiple roots, we will group terms in (2.4) in the following way. Assume that roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  coincide on a set



contained in some  $\mathcal{M}$ , that is  $\mathcal{M} \supset \{\tau_1(\xi) = \dots = \tau_L(\xi)\}$ . For  $\varepsilon > 0$ , we define  $\mathcal{M}^\varepsilon := \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}) < \varepsilon\}$ . Choose  $\varepsilon > 0$  so that these roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  do not intersect with any of the other roots  $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$  in  $\mathcal{M}^\varepsilon$ . If different numbers of roots intersect in different sets, we can apply the following theorem to such sets one by one. We note that by the strict hyperbolicity  $\mathcal{M}^\varepsilon$  is bounded. Here we will estimate the sum

$$\int_{\mathcal{M}^\varepsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi. \quad (2.8)$$

**Theorem 2.2.** *Let the sum (2.4) be the solution to the Cauchy problem (2.1). Assume that roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  coincide in a set contained in  $\mathcal{M}$  and do not intersect other roots in the set  $\mathcal{M}^\varepsilon$ . Let  $\chi \in C_0^\infty(\mathcal{M}^\varepsilon)$ . Assume that there exists  $\delta > 0$  such that  $\text{Im } \tau_k(\xi) \geq \delta$  for all  $\xi \in \mathcal{M}^\varepsilon$  and  $k = 1, \dots, L$ .*

*Then for all  $t \geq 0$  we have*

$$\left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\varepsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^p},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ .

Thus, if characteristic roots are separated from the real axis on the support of some  $\chi \in C_0^\infty(\mathbb{R}^n)$ , we can separate the solution (2.4) into groups of multiple roots for which the  $L^p - L^q$  norms still decay exponentially as stated in Theorem 2.2. We also note that since  $\mathcal{M}^\varepsilon$  is bounded, assumption (ii) of Theorem 2.1 is automatically satisfied and, therefore, it is omitted in the formulation of Theorem 2.2. Theorem 2.2 will be proved in Section 7.2.

## 2.2 Roots with non-degeneracies

The following case that we consider is the one of roots satisfying certain non-degeneracy conditions. These may be conditions on the Hessian, convexity conditions, or simply the information on the index of the corresponding level surfaces. In this section we will give the corresponding statements. We always assume the stability condition (2.2) but no longer assume that roots are separated from the real axis.

First we state the result for phases with the non-degenerate Hessian. The behavior depends on critical points  $\xi^0$  with  $\nabla \tau(\xi^0) = 0$  and the behavior of the Hessian at such points. As usual, we say that the critical point  $\xi^0$  is non-degenerate if the Hessian  $\text{Hess } \tau(\xi^0)$  is non-degenerate.

**Theorem 2.3.** *Let  $U \subset \mathbb{R}^n$  be a bounded open set, and let  $\tau : U \rightarrow \mathbb{C}$  be smooth and such that  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in U$ . Assume that there are some constants  $C_0$  and  $M$  such that  $|\det \text{Hess } \tau(\xi)| \geq C_0(1 + |\xi|)^{-M}$  for all  $\xi \in U$ . Let  $\chi \in S_{1,0}^0(\mathbb{R}^n)$  be such that  $\chi = 0$  outside  $U$  and let  $a \in S_{1,0}^{-\mu}(U)$ .*

Assume that  $\tau$  has only one non-degenerate critical point in  $U$ , and that  $U$  is sufficiently small. Then there is a constant  $C > 0$  independent of the position of  $U$  such that for all  $t \geq 0$  we have

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_p}}, \quad (2.9)$$

with  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $N_p = \frac{M}{2}(\frac{1}{p} - \frac{1}{q}) - \mu$ .

For example, the case of the Klein–Gordon equation corresponds to  $M = n + 2$  in this theorem. If we work with a fixed bounded set  $U$ , the  $\|f\|_{W_p^{N_p}}$  norm on the right hand side of (2.9) can be replaced by  $\|f\|_{L^p}$ . However, since we may also want to have estimate (2.9) uniform over such  $U$  (allowing it to move to infinity while remaining to be of the same size), we have the Sobolev norm in (2.9). From this point of view, we assume that  $a$  behaves as a symbol in  $U$  – the meaning is that if the symbolic constants here are uniform over the position of  $U$ , then also the constant in (2.9) is uniform over such  $a$  and  $U$ .

The condition that critical points are isolated and therefore can be localised by different sets  $U$  may follow from certain properties of  $\tau$  and will be discussed in Section 6.5, in particular see Lemma 6.7 and remarks after it. If, in addition, we take the size of  $U$  uniform, say of volume bounded by one, then constant  $C$  in (2.9) is also uniform over all such sets  $U$ . We may also assume that if  $\xi^0$  is a critical point of  $\tau$ , then  $\text{Im } \tau(\xi^0) = 0$ . Otherwise we would have  $\text{Im } \tau(\xi^0) > 0$  and so Theorem 2.1 would actually give the exponential decay rate. The proof of this theorem is based on the stationary phase method and will be given in Section 6.5. If we apply different versions of the stationary phase method under different conditions, we can reach different conclusions here. For example, we also have:

**Theorem 2.4.** *Let  $U \subset \mathbb{R}^n$  be a bounded open and let  $\tau : U \rightarrow \mathbb{C}$  be smooth and such that  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in U$ . Let  $\chi \in S_{1,0}^0(\mathbb{R}^n)$  be such that  $\chi = 0$  outside  $U$  and let  $a \in S_{1,0}^{-\mu}(U)$ . Assume that  $\tau$  has only one critical point  $\xi^0$  in  $U$ , and that  $U$  is sufficiently small.*

*Suppose that there are constants  $C_0, M > 0$  independent of the size and position of  $U$  and of  $\xi^0$ , with the following conditions. Suppose that  $\text{rank Hess } \tau(\xi^0) = k$ , that this rank is attained on an  $k \times k$  submatrix  $A(\xi^0)$  and that  $|\det A(\xi^0)| \geq C_0(1 + |\xi^0|)^{-M}$ . Then for all  $t \geq 0$  we have*

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{k}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_p}},$$

with  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $N_p = \frac{M}{2}(\frac{1}{p} - \frac{1}{q}) - \mu$ .

The proof of this theorem is similar to the proof of Theorem 2.3 once we restrict to the set of  $k$  variables (possibly after a suitable change) on which the rank of the Hessian is attained on  $A(\xi^0)$ .

This result can be improved dependent on further properties of  $A(\xi^0)$ . For example, if  $\text{rank } A(\xi^0) = n - 1$  and this is attained on variables  $\xi_1, \dots, \xi_{n-1}$ , the analysis reduces to the behaviour of the oscillatory integral with respect to  $\xi_n$ . If the  $l$ -th derivative of the phase with respect to  $\xi_n$  is non-zero, we get an additional decay by  $t^{-1/l}$ . This follows from the stationary phase method, see, for example Hörmander [Hör83a, Section 7.7], or from an appropriate use of van der Corput lemma. We will not formulate further statements here since they are quite straightforward.

The next theorem is an estimate of oscillatory integrals with real-valued phases under convexity condition. It will be shown in Proposition 3.8 (see also Proposition 6.16) that for large  $\xi$  characteristic roots of the Cauchy problem (2.1) satisfy assumptions of these theorems given below, if the homogeneous roots of the principal part satisfy them. The convexity condition is weaker than (but does not contain) the condition that the Hessian of  $\tau$  is positive definite and the result can be compared with Theorem 2.3, dependent on suitable properties of roots.

Let us first give the necessary definitions. Given a smooth function  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\lambda \in \mathbb{R}$ , set

$$\Sigma_\lambda \equiv \Sigma_\lambda(\tau) := \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}.$$

In the case where  $\tau(\xi)$  is homogeneous of order 1 and  $\tau \in C^\infty(\mathbb{R}^n \setminus \{0\})$ , we will also write  $\Sigma_\tau := \Sigma_1(\tau)$ —for such  $\tau$ , we then have  $\Sigma_\lambda(\tau) = \lambda \Sigma_\tau$ . There should be no confusion in this notation since we always reserve letters  $\phi, \tau$  for phases and  $\lambda$  for the real number.

**Definition 2.5.** *A smooth function  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to satisfy the convexity condition if surface  $\Sigma_\lambda$  is convex for each  $\lambda \in \mathbb{R}$ . Note that the empty set and the point set are considered to be convex.*

If the Gaussian curvatures of  $\Sigma_\lambda$  never vanish,  $\Sigma_\lambda$  is automatically convex (the converse is not true). This curvature condition corresponds to the case  $k = n - 1$  in Theorem 2.4. Another important notion is that of the *maximal order of contact* of a hypersurface, similar to the one in Section 1.2:

**Definition 2.6.** *Let  $\Sigma$  be a hypersurface in  $\mathbb{R}^n$  (i.e. a manifold of dimension  $n - 1$ ); let  $\sigma \in \Sigma$ , and denote the tangent plane at  $\sigma$  by  $T_\sigma$ . Now let  $P$  be a 2-dimensional plane containing the normal to  $\Sigma$  at  $\sigma$  and denote the order of the contact between the line  $T_\sigma \cap P$  and the curve  $\Sigma \cap P$  by  $\gamma(\Sigma; \sigma, P)$ . Then set*

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_P \gamma(\Sigma; \sigma, P).$$

## Examples 2.7.

- (a)  $\gamma(\mathbb{S}^n) = 2$ , as  $\gamma(\mathbb{S}^n; \sigma, P) = 2$  for all  $\sigma \in \mathbb{S}^n$  and all planes  $P$  containing  $\sigma$  and the origin.
- (b) If  $\varphi_l(\xi)$  is a characteristic root of an  $m^{\text{th}}$  order homogeneous strictly hyperbolic constant coefficient operator, then  $\gamma(\Sigma_{\varphi_l}) \leq m$ ; see [Sug96] for a proof of this.

Now we can formulate the corresponding theorem.

**Theorem 2.8.** *Suppose  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the convexity condition and let  $\chi \in C^\infty(\mathbb{R}^n)$ ; furthermore, on  $\text{supp } \chi$ , we assume:*

(i) *for all multi-indices  $\alpha$  there exists a constant  $C_\alpha > 0$  such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$

(ii) *there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\tau(\xi)| \geq C|\xi|$ ;*

(iii) *there exists a constant  $C > 0$  such that  $|\partial_\omega \tau(\lambda\omega)| \geq C$  for all  $\omega \in \mathbb{S}^{n-1}$ ,  $\lambda > 0$ ; in particular,  $|\nabla \tau(\xi)| \geq C$  for all  $\xi \in \mathbb{R}^n \setminus \{0\}$ ;*

(iv) *there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,*

$$\frac{1}{\lambda} \Sigma_\lambda(\tau) \equiv \frac{1}{\lambda} \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0).$$

Also, set  $\gamma := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau))$  and assume this is finite. Let  $a_j = a_j(\xi) \in S_{1,0}^{-j}$  be a symbol of order  $-j$  of type  $(1, 0)$  on  $\mathbb{R}^n$ . Then for all  $t \geq 0$  we have the estimate

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_{p,j,t}}}, \quad (2.10)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$ , and the Sobolev order satisfies  $N_{p,j,t} \geq n(\frac{1}{p} - \frac{1}{q}) - j$  for  $0 \leq t < 1$ , and  $N_{p,j,t} \geq \left(n - \frac{n-1}{\gamma}\right)(\frac{1}{p} - \frac{1}{q}) - j$  for  $t \geq 1$ .

Theorem 2.8 will be proved in Section 6.6, where estimate (2.10) will follow by interpolation from the  $L^2 - L^2$  estimate combined with  $L^1 - L^\infty$  cases given in (6.11) for small  $t$ , and in (6.16) for large  $t$ . See those estimates also for the case of  $p = 1$  in estimate (2.10). The estimate for large times will follow from Theorem 4.8, which gives the  $L^\infty$ -estimate for the kernel of (2.10). As another consequence of Theorem 4.8, we will also have the following estimate:

**Corollary 2.9.** *Under conditions of Theorem 2.8 with  $\chi \equiv 1$ , assume that  $a \in C_0^\infty(\mathbb{R}^n)$ . Then for all  $x \in \mathbb{R}^n$  and  $t \geq 0$  we have the estimate*

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) d\xi \right| \leq C(1+t)^{-\frac{n-1}{\gamma}}. \quad (2.11)$$

In Proposition 3.8 we show that properties (i)–(iv) of Theorem 2.8 are satisfied for characteristic roots of  $L(D_t, D_x)$  in (2.1), while in Lemma 6.11 we will show that the index  $\gamma$  is also finite, both for large frequencies.

Now we turn to the case without convexity. As in the case of the homogeneous operators (see Introduction, Section 1.2) we introduce an analog of the order of contact also in the case where the convexity condition does not hold.

**Definition 2.10.** Let  $\Sigma$  be a hypersurface in  $\mathbb{R}^n$ ; set

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_P \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma),$$

where  $\gamma(\Sigma; \sigma, P)$  is as in Definition 2.6.

**Remark 2.11.**

- (a) When  $n = 2$ ,  $\gamma_0(\Sigma) = \gamma(\Sigma)$ ;
- (b) If  $p(\xi)$  is a polynomial of order  $m$ ,  $\Sigma = \{\xi \in \mathbb{R}^n : p(\xi) = 0\}$  is compact, and  $\nabla p(\xi) \neq 0$  on  $\Sigma$ , then  $\gamma_0(\Sigma) \leq \gamma(\Sigma) \leq m$ ; this is useful when applying the result below to hyperbolic differential equations and is proved in [Sug96].

**Theorem 2.12.** Suppose  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function. Let  $\chi \in C^\infty(\mathbb{R}^n)$ ; furthermore, on  $\text{supp } \chi$ , we assume:

- (i) for all multi-indices  $\alpha$  there exist constants  $C_\alpha > 0$  such that
$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$
- (ii) there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\tau(\xi)| \geq C|\xi|$ ;
- (iii) there exists a constant  $C > 0$  such that  $|\partial_\omega \tau(\lambda\omega)| \geq C$  for all  $\omega \in \mathbb{S}^{n-1}$  and  $\lambda > 0$ ;
- (iv) there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,

$$\frac{1}{\lambda} \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0).$$

Set  $\gamma_0 := \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\tau))$  and assume it is finite. Let  $a_j = a_j(\xi) \in S_{1,0}^{-j}$  be a symbol of order  $-j$  of type  $(1,0)$  on  $\mathbb{R}^n$ . Then for all  $t \geq 0$  we have the estimate

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_{p,j,t}}},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$ , and the Sobolev order satisfies  $N_{p,j,t} \geq n(\frac{1}{p} - \frac{1}{q}) - j$  for  $0 \leq t < 1$ , and  $N_{p,j,t} \geq \left(n - \frac{1}{\gamma_0}\right)(\frac{1}{p} - \frac{1}{q}) - j$  for  $t \geq 1$ .

The proof of Theorem 2.12 will be given in Section 6.7. As in the convex case, as a consequence of estimates for the kernel on Theorem 5.3, we also have the following statement:

**Corollary 2.13.** Under conditions of Theorem 2.12 with  $\chi \equiv 1$ , assume that  $a \in C_0^\infty(\mathbb{R}^n)$ . Then for all  $x \in \mathbb{R}^n$  and  $t \geq 0$  we have the estimate for the kernel:

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) d\xi \right| \leq C(1+t)^{-\frac{1}{\gamma_0}}.$$

Again, in Proposition 3.8 we show that properties (i)–(iv) of Theorem 2.12 are satisfied for characteristic roots of  $L(D_t, D_x)$  in (2.1), while in Lemma 6.14 we will show that the index  $\gamma_0$  is also finite, both for large frequencies.

As a corollary and an example of these theorems, we get the following possibilities of decay for parts of solutions with roots on the axis. We can use a cut-off function  $\chi$  to microlocalise around points with different qualitative behaviour (hence we also do not have to worry about Sobolev orders).

**Corollary 2.14.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set and let  $\tau : \Omega \rightarrow \mathbb{R}$  be a smooth real valued function. Let  $\chi \in C_0^\infty(\Omega)$ . Let us make the following choices of  $K(t)$ , depending on which of the following conditions are satisfied on  $\text{supp } \chi$ .*

- (1) *If  $\det \text{Hess } \tau(\xi) \neq 0$  for all  $\xi \in \Omega$ , we set  $K(t) = (1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$ .*
- (2) *If  $\text{rank Hess } \tau(\xi) = n-1$  for all  $\xi \in \Omega$ , we set  $K(t) = (1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$ .*
- (3) *If  $\tau$  satisfies the convexity condition with index  $\gamma$ , we set  $K(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$ .*
- (4) *If  $\tau$  does not satisfy the convexity condition but has non-convex index  $\gamma_0$ , we set  $K(t) = (1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$ .*

*Assume in each case that other assumptions of the corresponding Theorems 2.3–2.12 are satisfied. Let  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then for all  $t \geq 0$  we have*

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq CK(t) \|f\|_{L^p(\mathbb{R}^n)}.$$

We note that no derivatives appear in the  $L^p$ -norm of  $f$  because the support of  $\chi$  is bounded. In general, there are different ways to ensure the convexity condition for  $\tau$ . Thus, we can say that the principal part  $L_m$  of operator  $L(D_t, D_x)$  in (2.1) satisfies the convexity condition if all Hessians  $\varphi_l''(\xi)$ ,  $l = 1, \dots, m$ , are semi-definite for all  $\xi \neq 0$ . In this case it was shown by Sugimoto in [Sug94] that there exists a linear function  $\alpha(\xi)$  such that  $\tilde{\varphi}_l = \varphi_l + \alpha$  have convex level sets  $\Sigma(\tilde{\varphi}_l)$ , and we have  $\gamma(\Sigma(\tilde{\varphi}_l)) \leq 2 \left\lceil \frac{m}{2} \right\rceil$ . For large frequencies, perturbation arguments imply that the same must be true for  $\Sigma_\lambda(\tau_l)$ , for sufficiently large  $\lambda$ . If we now assume that  $\Sigma_\lambda(\tau_l)$  are also convex for small  $\lambda$ , then  $\tau_l$  will satisfy the convexity conditions. Alternatively, if they do not satisfy the convexity condition for small  $\lambda$ , we can cut-off this regions and analyse the decay rates by other methods developed in this paper.

## 2.3 Roots meeting the real axis

In this section we will present the results for characteristic roots (or phase functions) in the upper complex plane near the real axis, that become real at some point or in some set.

For  $\mathcal{M} \subset \mathbb{R}^n$ , denote  $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}) < \varepsilon\}$  as before. The largest number  $\nu \in \mathbb{N}$  such that  $\text{meas}(\mathcal{M}^\varepsilon) \leq C\varepsilon^\nu$  for all sufficiently small  $\varepsilon > 0$ , will be denoted by  $\text{codim } \mathcal{M}$ , and we will call it the codimension of  $\mathcal{M}$ .

We will say that *the root  $\tau_k$  meets the real axis at  $\xi^0$  with order  $s_k$*  if  $\text{Im } \tau_k(\xi^0) = 0$  and if there exists a constant  $c_0 > 0$  such that

$$c_0 |\xi - \xi^0|^{s_k} \leq \text{Im } \tau_k(\xi),$$

for all  $\xi$  sufficiently near  $\xi^0$ . Here we may recall that in (2.2) we already assumed  $\text{Im } \tau_k(\xi) \geq 0$  for all  $\xi$ .

More generally, if the root  $\tau_k$  meets the axis on the set  $Z_k = \{\xi \in \mathbb{R}^n : \text{Im } \tau_k(\xi) = 0\}$ , we will say that *it meets the axis with order  $s$*  if

$$c_0 \text{dist}(\xi, Z_k)^s \leq \text{Im } \tau_k(\xi).$$

We will localise around each connected component of  $Z_k$ , e.g. around each point of  $Z_k$ , if it is a union of isolated points. As usual, when we talk about multiple roots intersecting in a set  $\mathcal{M}$ , we adopt the terminology introduced in Section 2.1. Since we are dealing with strictly hyperbolic equations, roots can meet each other only for bounded frequencies, so we may assume that set  $\mathcal{M}$  is bounded.

**Theorem 2.15.** *Assume that the characteristic roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  intersect in the  $C^1$  set  $\mathcal{M}$  of codimension  $\ell$ . Assume also that they meet the real axis in  $\mathcal{M}$  with the finite orders  $\leq s$ , i.e. that*

$$c_0 \text{dist}(\xi, \mathcal{M})^s \leq \text{Im } \tau_k(\xi),$$

for some  $c_0 > 0$  and all  $k = 1, \dots, L$ . Assume that (2.4) is the solution of the Cauchy problem (2.1) and we look at its part (2.8). Let  $\chi \in C_0^\infty(\mathcal{M}^\varepsilon)$  for sufficiently small  $\varepsilon > 0$ . Then for all  $t \geq 0$  we have

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\varepsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^q(\mathbb{R}_x^n)} \\ \leq C(1+t)^{-\frac{\ell}{s} \left( \frac{1}{p} - \frac{1}{q} \right) + L-1} \|f\|_{L^p}, \quad (2.12) \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ .

We assume  $\varepsilon > 0$  to be small enough to make sure that the type of behaviour assumed in the theorem is the only one that takes place in  $\mathcal{M}^\varepsilon$ . In the complement of  $\mathcal{M}^\varepsilon$  we may use other theorems to analyse the decay rate. Moreover, we assume that set  $\mathcal{M}$  is  $C^1$ . In fact, it is usually Lipschitz, so in order to avoid to go into depth about its structure and existence of almost everywhere differentiable coordinate systems, we make the technical  $C^1$  assumption. The proof of Theorem 2.15 will be given in Section 7.3.

Let us now give a special case of this theorem where simple roots meet the axis at a point, so that we have  $L = 1$  and  $\ell = n$ . The following statement is also global in frequency, so we have the result in Sobolev spaces.



**Theorem 2.16.** Consider the  $m^{\text{th}}$  order strictly hyperbolic Cauchy problem (2.1) for operator  $L(D_t, D_x)$ , with initial data  $f_j \in W_p^{N_p+|\alpha|+r-j}$ , for  $j = 0, \dots, m-1$ , where  $1 \leq p \leq 2$  and  $2 \leq q \leq \infty$  are such that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $r \geq 0$  and  $\alpha$  is a multi-index. We assume that the Sobolev index  $N_p$  satisfies  $N_p \geq n(\frac{1}{p} - \frac{1}{q})$  for  $1 < p \leq 2$  and  $N_1 > n$  for  $p = 1$ .

Assume that the characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  of  $L(\tau, \xi) = 0$  satisfy  $\text{Im } \tau_k \geq 0$  for all  $k$ , and also the following conditions:

(H1) for all  $k = 1, \dots, m$ , we have

$$\liminf_{|\xi| \rightarrow \infty} \text{Im } \tau_k(\xi) > 0;$$

(H2) for each  $\xi^0 \in \mathbb{R}^n$  there is at most one index  $k$  for which  $\text{Im } \tau_k(\xi^0) = 0$  and there exists a constant  $c > 0$  such that

$$|\xi - \xi^0|^s \leq c \text{Im } \tau_k(\xi),$$

for  $\xi$  in some neighbourhood of  $\xi^0$ . Assume also that there are finitely many points  $\xi^0$  with  $\text{Im } \tau_k(\xi^0) = 0$ .

Then the solution  $u = u(t, x)$  to Cauchy problem (2.1) satisfies the following estimate for all  $t \geq 0$ :

$$\|D_t^r D_x^\alpha u(t, \cdot)\|_{L^q} \leq C_{\alpha, r} (1+t)^{-\frac{n}{s}(\frac{1}{p} - \frac{1}{q})} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}. \quad (2.13)$$

Theorem 2.16 is proved in Section 6.11, where we will also give microlocal versions of this result around points  $\xi^0$  from hypothesis (H2). In the complement of such points, we have roots separated from the real axis, so we get the exponential decay from Theorems 2.1 and 2.2. Moreover, in the exponential decay zone we may have different versions of the estimate, for example we can use estimate (2.7) there instead of (2.6). As a special case, such estimate together with (2.15) below (used with  $s = s_1 = 2$ ), we improve the indices in Sobolev spaces over  $L^2$  for the dissipative wave equation in (1.7) and (1.8) compared to [Mat77].

If conditions of Theorem 2.16 hold only with  $\xi^0 = 0$ , namely if  $\text{Im } \tau_k(\xi^0) = 0$  implies  $\xi^0 = 0$ , we will call the polynomial  $L(\tau, \xi)$  *strongly stable*. Such polynomials will be discussed in more detail in applications in Section 8.5. Now we will give some improvements of (2.13) under additional assumptions on the roots:

**Remark 2.17.** The order of time decay in Theorem 2.16 may be improved in the following cases, if we make additional assumptions. If, in addition, we assume that  $\text{Im } \tau_k(\xi^0) = 0$  in (H2) implies that  $\xi^0 = 0$ , then we actually get the estimate

$$\left\| D_t^r D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}_x^n)} \leq C (1+t)^{-\frac{n}{s}(\frac{1}{p} - \frac{1}{q}) - \frac{|\alpha|}{2}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}},$$

where here and further in this remark  $N_p$  is as in Theorem 2.16.

Now, assume further that for all  $\xi^0$  in (H2) we also have the estimate

$$|\tau_k(\xi)| \leq c_1 |\xi - \xi^0|^{s_1}, \quad (2.14)$$

with some constant  $c_1 > 0$ , for all  $\xi$  sufficiently close to  $\xi^0$ .

If we have that  $\text{Im } \tau_k(\xi^0) = 0$  in (H2) implies that we have (2.14) around such  $\xi^0$ , then we actually get

$$\left\| D_t^r D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\left(\frac{n}{s}\right)\left(\frac{1}{p}-\frac{1}{q}\right) - \frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}.$$

And finally, assume that for all  $\xi^0$  such that  $\text{Im } \tau_k(\xi^0) = 0$  in (H2), we also have  $\xi^0 = 0$  and (2.14) around such  $\xi^0$ . Then we actually get

$$\left\| D_t^r D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right) - \frac{|\alpha|}{s} - \frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}. \quad (2.15)$$

Estimate (2.15) with  $s = s_1 = 2$  gives the decay estimate for the dissipative wave equation in (1.7). The proof of this remark is given in Remark 6.19.

Moreover, there are other possibilities of *multiple roots intersecting each other while lying entirely on the real axis*. For example, this is the case for the wave equation or for more general equations with homogeneous symbols, when several roots meet at the origin. In this case roots always lie on the real axis, but they become irregular at the point of multiplicity, which is the origin for homogeneous roots. In the case when lower order terms are presents, characteristics roots are not homogeneous in general, so we can not eliminate time from the estimates as was done in Section 1.2. It means that we have to look at the structure of such multiple points by making cut-offs around them and studying their structure in more detail. In particular, there is an interaction between low frequencies and large times, which does not take place for homogeneous symbols. The detailed discussion of this topic and corresponding decay rates will be determined in Section 7.4.

## 2.4 Application to the Cauchy problem

Putting together theorems from previous sections we obtain the following conclusion about solutions to the Cauchy problem (2.1). We will first formulate the following general result collecting statements of previous sections, and then will explain how this result can be used.

**Theorem 2.18.** *Suppose  $u = u(t, x)$  is the solution of the  $m^{\text{th}}$  order linear, constant coefficient, strictly hyperbolic Cauchy problem (2.1). Denote the characteristic roots of the operator by  $\tau_1(\xi), \dots, \tau_m(\xi)$ , and assume that  $\text{Im } \tau_k(\xi) \geq 0$  for all  $k = 1, \dots, n$ , and all  $\xi \in \mathbb{R}^n$ .*

*We introduce two functions,  $K^{(l)}(t)$  and  $K^{(b)}(t)$ , which take values as follows:*

- I. Consider the behaviour of each characteristic root,  $\tau_k(\xi)$ , in the region  $|\xi| \geq M$ , where  $M$  is a large enough real number. The following table gives values for the function  $K_k^{(l)}(t)$  corresponding to possible properties of  $\tau_k(\xi)$ ; if  $\tau_k(\xi)$  satisfies more than one, then take  $K_k^{(l)}(t)$  to be function that decays the slowest as  $t \rightarrow \infty$ .

Location of $\tau_k(\xi)$	Additional Property	$K_k^{(l)}(t)$
away from real axis		$e^{-\delta t}$ , some $\delta > 0$
on real axis	$\det \text{Hess } \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	$\text{rank Hess } \tau_k(\xi) = n-1$	$(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition $\gamma$	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, $\gamma_0$	$(1+t)^{-\frac{1}{\gamma_0}}$

Then take  $K^{(l)}(t) = \max_{k=1, \dots, n} K_k^{(l)}(t)$ .

- II. Consider the behaviour of the characteristic roots in the bounded region  $|\xi| \leq M$ ; again, take  $K^{(b)}(t)$  to be the maximum (slowest decaying) function for which there are roots satisfying the conditions in the following table:

Location of Root(s)	Properties	$K^{(b)}(t)$
away from axis	no multiplicities $L$ roots coinciding	$e^{-\delta t}$ , some $\delta > 0$ $(1+t)^L e^{-\delta t}$
on axis, no multiplicities *	$\det \text{Hess } \tau_k(\xi) \neq 0$ convexity condition $\gamma$ no convexity condition, $\gamma_0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$ $(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$ $(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$
on axis, multiplicities*, **	$L$ roots coincide on set of codimension $\ell$	$(1+t)^{L-1-\ell}$
meeting axis with finite order $s$	$L$ roots coincide on set of codimension $\ell$	$(1+t)^{L-1-\frac{\ell}{s}(\frac{1}{p}-\frac{1}{q})}$

\* These two cases of roots lying on the real axis require some additional regularity assumptions; see corresponding microlocal statements for details.

\*\* This is the  $L^1 - L^\infty$  rate in a shrinking region; see Proposition 7.9 for details. For different types of  $L^2$  estimates see Section 7.4, and then interpolate.

Then, with  $K(t) = \max(K^{(b)}(t), K^{(l)}(t))$ , the following estimate holds:

$$\|D_x^\alpha D_t^r u(t, \cdot)\|_{L^q} \leq C_{\alpha, r} K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}},$$

where  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $N_p = N_p(\alpha, r)$  is a constant depending on  $p, \alpha$  and  $r$ .

The scheme of the proof of this theorem and precise relations to microlocal theorems of previous sections will be given in Section 2.5. However, let us now briefly explain how to understand this theorem. Since the decay rates do depend on the behaviour of characteristic roots in different regions and theorems from previous sections determine the corresponding rates, in Theorem 2.18 we single out properties which determine the final decay rate. Since the same characteristic root, say  $\tau_k$ , may exhibit different properties in different regions, we look at the corresponding rates  $K^{(b)}(t)$ ,  $K^{(l)}(t)$  under each possible condition and then take the slowest one for the final answer. The value of the Sobolev index  $N_p = N_p(\alpha, r)$  depends on the regions as well, and it can be found from microlocal statements of previous sections for each region.

In conditions of Part I of the theorem, it can be shown by the perturbation arguments that only three cases are possible for large  $\xi$ , namely, the characteristic root may be uniformly separated from the real axis, it may lie on the axis, or it may converge to the real axis at infinity. If, for example, the root lies on the axis and, in addition, it satisfies the convexity condition with index  $\gamma$ , we get the corresponding decay rate  $K^{(l)}(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$ . Indices  $\gamma$  and  $\gamma_0$  in the tables are defined as the maximum of the corresponding indices  $\gamma(\Sigma_\lambda)$  and  $\gamma_0(\Sigma_\lambda)$ , respectively, where  $\Sigma_\lambda = \{\xi : \tau_k(\xi) = \lambda\}$ , over all  $k$  and over all  $\lambda$ , for which  $\xi$  lies in the corresponding region. At present, we do not have examples of characteristic roots tending to the real axis for large frequencies while remaining in the open upper half of the complex plane, so we do not give any estimates for this case in Theorem 2.18. However, in Section 6.8 we will still discuss what happens in this case.

The statement in Part II is more involved since we may have multiple roots intersecting on rather irregular sets. The number  $L$  of coinciding roots corresponds to the number of roots which actually contribute to the loss of regularity. For example, operator  $(\partial_t^2 - \Delta)(\partial_t^2 - 2\Delta)$  would have  $L = 2$  for both pairs of roots  $\pm|\xi|$  and  $\pm\sqrt{2}|\xi|$ , intersecting at the origin. Meeting the axis with finite order  $s$  means that we have the estimate

$$\text{dist}(\xi, Z_k)^s \leq c |\text{Im } \tau_k(\xi)| \quad (2.16)$$

for all the intersecting roots, where  $Z_k = \{\xi : \text{Im } \tau_k(\xi) = 0\}$ . In Part II of Theorem 2.18, the condition that  $L$  roots meet the axis with finite order  $s$  on a set of codimension  $\ell$  means that all these estimates hold and that there is a  $(C^1)$  set  $\mathcal{M}$  of codimension  $\ell$  such that  $Z_k \subset \mathcal{M}$  for all corresponding  $k$  (see Theorem 2.15 for details). In Theorem 2.16 we discuss the special case of a single root  $\tau_k$  meeting the axis at a point  $\xi_0$  with order  $s$ , which means that  $\text{Im } \tau_k(\xi_0) = 0$  and that we have the estimate  $|\xi - \xi_0|^s \leq c |\text{Im } \tau_k(\xi)|$ . In fact, under certain conditions an improvement in this part of the estimates is possible, see Theorem 2.16 and Remark 2.17.

In Part II of the theorem, condition \*\* is formulated in the region of the size decreasing with time: if we have  $L$  multiple roots which coincide on the real axis on a set  $\mathcal{M}$  of codimension  $\ell$ , we have an estimate

$$|u(t, x)| \leq C(1+t)^{L-1-\ell} \sum_{l=0}^{m-1} \|f_l\|_{L^1}, \quad (2.17)$$

if we cut off the Fourier transforms of the Cauchy data to the  $\epsilon$ -neighbourhood  $\mathcal{M}^\epsilon$  of  $\mathcal{M}$  with  $\epsilon = 1/t$ . Here we may relax the definition of the intersection above and say that if  $L$  roots coincide in a set  $\mathcal{M}$ , then they coincide on a set of codimension  $\ell$  if the measure of the  $\epsilon$ -neighborhood  $\mathcal{M}^\epsilon$  of  $\mathcal{M}$  satisfies  $|\mathcal{M}^\epsilon| \leq C\epsilon^\ell$  for small  $\epsilon > 0$ ; here  $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}) \leq \epsilon\}$ . The estimate (2.17) follows from the procedure described in Section 7.1 of the resolution of multiple roots, and details and proof of estimate (2.17) are given in Section 7.4, especially in Proposition 7.9.

We can then combine this with the remaining cases outside of this neighborhood, where it is possible to establish decay by different arguments. In particular, this is the case of homogeneous equations with roots intersecting at the origin. However, one sometimes needs to introduce special norms to handle  $L^2$ -estimates around the multiplicities. Details of this are given in the  $L^2$  part of Section 7.4.2, in particular in Proposition 7.5. Finally, in the case of a simple root we may set  $L = 1$ , and  $\ell = n$ , if it meets the axis at a point.

## 2.5 Schematic of method

Let us briefly explain some ideas behind the reduction of Theorem 2.18 to the proceeding theorems. The realisation of the steps below will be done in Sections 6 and 7.

**Step 1:** Representation of the solution.

Using the Fourier transform in  $x$ , this reduces the problem to studying time-dependent oscillatory integrals, at least for frequencies with no multiplicities. In the case near multiplicities we will introduce a special procedure to deal with them in Section 7.

**Step 2:** Division of the integral.

We reduce the problem to several microlocal cases using suitable cut-off functions. The problem is divided into studying the behaviour of the characteristic roots in three regions of the phase space—large  $|\xi|$ , bounded  $|\xi|$  away from multiplicities of roots and bounded  $|\xi|$  in a neighbourhood of multiplicities.

**Step 3:** Interpolation reduces problem to finding  $L^1 - L^\infty$  and  $L^2 - L^2$  estimates.

**Step 4:** Large  $|\xi|$ :

- root separated from the real axis (Theorem 2.1);
- root lying on the real axis (Theorems 2.4–2.12).

**Step 5:** Bounded  $|\xi|$ , away from multiplicities:

- root away from the real axis (Theorem 2.1);
- root meeting the real axis with finite order (Theorem 2.16);
- root lying on the real axis (Theorems 2.4–2.12).

**Step 6:** Bounded  $|\xi|$ , around multiplicities of roots:

- all intersecting roots away from the real axis (Theorem 2.2);
- all intersecting roots lie on the real axis around the multiplicity (Section 7.4);
- all intersecting roots meet the real axis with finite order (Theorem 2.15);
- one or more of the roots meets the real axis with infinite order (similar to Theorems 2.4–2.12).

## 2.6 Strichartz estimates and nonlinear problems

Let us denote by  $\kappa_{p,q}(L(D_t, D_x))$  the time decay rate for the Cauchy problem (2.1), so that function  $K(t)$  from Theorem 2.18 satisfies  $K(t) \simeq t^{-\kappa_{p,q}(L)}$  for large  $t$ . Thus, for polynomial decay rates, we have

$$\kappa_{p,q}(L) = -\lim_{t \rightarrow \infty} \frac{\ln K(t)}{\ln t}. \quad (2.18)$$

We will also abbreviate the important case  $\kappa(L) = \kappa_{1,\infty}(L)$  since by interpolation we have  $\kappa_{p,p'} = \kappa_{2,2} \frac{2}{p} + \kappa_{1,\infty}(\frac{1}{p} - \frac{1}{p'})$ ,  $1 \leq p \leq 2$ . These indices  $\kappa(L)$  and  $\kappa_{p,p'}(L)$  of operator  $L(D_t, D_x)$  will be responsible for the decay rate in the Strichartz estimates for solutions to (2.1), and for the subsequent well-posedness properties of the corresponding semilinear equation which are discussed below.

In order to present an application to nonlinear problems let us first consider the inhomogeneous equation

$$\begin{cases} L(D_t, D_x)u = f, & t > 0, \\ D_t^l u(0, x) = 0, & l = 0, \dots, m-1, \ x \in \mathbb{R}^n, \end{cases} \quad (2.19)$$

with  $L(D_t, D_x)$  as in (1.1). By the Duhamel's formula the solution can be expressed as

$$u(t) = \int_0^t E_{m-1}(t-s)f(s)ds, \quad (2.20)$$

where  $E_{m-1}$  is given in (2.4). Let  $\kappa = \kappa_{p,p'}(L)$  be the time decay rate of operator  $L$ , determined by Theorem 2.18 and given in (2.18). Then Theorem 2.18 implies that we have estimate

$$\|E_{m-1}(t)g\|_{W_{p'}^s} \leq C(1+t)^{-\kappa}\|g\|_{W_p^s}.$$

Together with (2.20) this implies

$$\|u(t)\|_{W_{p'}^s(\mathbb{R}_x^n)} \leq C \int_0^t (t-s)^{-\kappa} \|f(s)\|_{W_p^s} ds \leq C|t|^{-\kappa} * \|f(t)\|_{W_p^s}.$$

By the Hardy–Littlewood–Sobolev theorem this is  $L^q(\mathbb{R}) - L^{q'}(\mathbb{R})$  bounded if  $1 < q < 2$  and  $1 - \kappa = \frac{1}{q} - \frac{1}{q'}$ . Therefore, this implies the following Strichartz estimate:

**Theorem 2.19.** *Let  $\kappa_{p,p'}$  be the time decay rate of the operator  $L(D_t, D_x)$  in the Cauchy problem (2.19). Let  $1 < p, q < 2$  be such that  $1/p + 1/p' = 1/q + 1/q' = 1$  and  $1/q - 1/q' = 1 - \kappa_{p,p'}$ . Let  $s \in \mathbb{R}$ . Then there is a constant  $C$  such that the solution  $u$  to the Cauchy problem (2.19) satisfies*

$$\|u\|_{L^{q'}(\mathbb{R}_t, W_{p'}^s(\mathbb{R}_x^n))} \leq C \|f\|_{L^q(\mathbb{R}_t, W_p^s(\mathbb{R}_x^n))},$$

for all data right hand side  $f = f(t, x)$ .

By the standard iteration method we obtain the well-posedness result for the following semilinear equation

$$\begin{cases} L(D_t, D_x)u = F(t, x, u), & t > 0, \\ D_t^l u(0, x) = f_l(x), & l = 0, \dots, m-1, x \in \mathbb{R}^n. \end{cases} \quad (2.21)$$

**Theorem 2.20.** *Let  $\kappa_{p,p'}$  be the time decay index of the operator  $L(D_t, D_x)$  in the Cauchy problem (2.21). Let  $p, q$  be such that  $1/p + 1/p' = 1/q + 1/q' = 1$  and  $1/q - 1/q' = 1 - \kappa_{p,p'}$ . Let  $s \in \mathbb{R}$ .*

*Assume that for any  $v \in L^{q'}(\mathbb{R}_t, W_{p'}^s(\mathbb{R}_x^n))$ , the nonlinear term satisfies  $F(t, x, v) \in L^q(\mathbb{R}_t, W_p^s(\mathbb{R}_x^n))$ . Moreover, assume that for every  $\varepsilon > 0$  there exists a decomposition  $-\infty = t_0 < t_1 < \dots < t_k = +\infty$  such that the estimates*

$$\|F(t, x, u) - F(t, x, v)\|_{L^q(I_j, W_p^s(\mathbb{R}_x^n))} \leq \varepsilon \|u - v\|_{L^{q'}(I_j, W_{p'}^s(\mathbb{R}_x^n))}$$

*hold for the intervals  $I_j = (t_j, t_{j+1})$ ,  $j = 0, \dots, k-1$ .*

*Finally, assume that the solution of the corresponding homogeneous Cauchy problem is in the space  $L^{q'}(\mathbb{R}_t, W_{p'}^s(\mathbb{R}_x^n))$ .*

*Then the semilinear Cauchy problem (2.21) has a unique solution in the space  $L^{q'}(\mathbb{R}_t, W_{p'}^s(\mathbb{R}_x^n))$ .*

### 3 Properties of hyperbolic polynomials

In order to study the solution  $u(t, x)$  to (1.1), we must first know some properties of the characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$ . Naturally, we do not have explicit formulae for the roots, unlike in the cases of the dissipative wave equation and the Klein–Gordon equation (i.e. for second order equations), but we do know some properties for the roots of the principal symbol. For general hyperbolic operators, the roots  $\varphi_1(\xi), \dots, \varphi_m(\xi)$  of the characteristic polynomial of the *principal part* are homogeneous functions of order 1 since the principal part is homogeneous. Furthermore, for strictly hyperbolic polynomials these roots are distinct when  $\xi \neq 0$ . Since these two properties are very useful when studying homogeneous (strictly) hyperbolic equations, it is useful to know whether the characteristic roots of the full equation,  $\tau_1(\xi), \dots, \tau_m(\xi)$ , have similar properties. Indeed, if we regard the full equation as a perturbation of the principal part by lower order terms, we can show that similar properties hold for large  $|\xi|$ ; these results are the focus of this section. In the outline of the method in Section 2.5, we subdivided the phase space into large  $|\xi|$  and bounded  $|\xi|$ , and it is these properties that motivate this step.



### 3.1 General properties

First, we give some properties of general polynomials which are useful to us. For constant coefficient polynomials, the following result holds:

**Lemma 3.1.** *Consider the polynomial over  $\mathbb{C}$  with complex coefficients*

$$z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m = \prod_{k=1}^m (z - z_k).$$

*If there exists  $M > 0$  such that  $|c_j| \leq M^j$  for each  $j = 1, \dots, m$ , then  $|z_k| \leq 2M$  for all  $k = 1, \dots, m$ .*

*Proof.* Assume that  $|z| > 2M$ . Then

$$\begin{aligned} |z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m| &\geq |z|^m \left( 1 - \frac{|c_1|}{|z|} - \cdots - \frac{|c_{m-1}|}{|z|^{m-1}} - \frac{|c_m|}{|z|^m} \right) \\ &\geq (2M)^m (1 - 2^{-1} - \cdots - 2^{-(m-1)} - 2^{-m}) > 0. \end{aligned}$$

That is, no zero of the polynomial lies outside of the ball about the origin of radius  $2M$ ; hence  $|z_k| \leq 2M$  for each  $k = 1, \dots, m$ .  $\square$

**Remark 3.2.** *If we replace the hypothesis  $|c_j| \leq M^j$  by  $|c_j| \leq M$  for each  $j = 1, \dots, m$ , then by a similar argument we obtain that  $|z_k| \leq \max\{2, 2M\}$ . The quantity  $\max\{2, 2M\}$  appears because we need  $M \geq 1$  for the sum on the right hand side to be positive.*

For general polynomials with variable coefficients, we have continuous dependence of roots on coefficients (we give an independent proof of this result here for the sake of completeness and for referencing, but analogue of this result can be found in many monographs dealing with hyperbolic polynomials).

**Lemma 3.3.** *Consider the  $m^{\text{th}}$  order polynomial with coefficients depending on  $\xi \in \mathbb{R}^n$*

$$p(\tau, \xi) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_m(\xi).$$

*If each of the coefficient functions  $a_j(\xi)$ ,  $j = 1, \dots, m$ , is continuous in  $\mathbb{R}^n$  then each of the roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  with respect to  $\tau$  of  $p(\tau, \xi) = 0$  is also continuous in  $\mathbb{R}^n$ .*

*Proof.* Define  $\rho : \mathbb{C}^m \rightarrow \mathbb{C}^m$  by  $\rho(z_1, \dots, z_m) = (c_1, \dots, c_m)$  where the  $c_j$  satisfy

$$z^m + c_1 z^{m-1} + \cdots + c_m = \prod_{j=1}^m (z - z_j).$$

By the fundamental theorem of algebra  $\rho$  is invertible (but the inverse is not unique modulo permutation of roots), and, moreover,  $\rho$  is:

- (a) surjective by the Fundamental Theorem of Algebra;

(b) continuous since each of the  $c_j$  may be written as polynomials of the  $z_j$  (by the Viète formulae);

(c) proper (that is, the preimage of each compact set is compact) by Remark 3.2;

properties (b) and (c) imply that  $\rho$  is a closed mapping.

Now, fix  $\xi^0 \in \mathbb{R}^n$ . For any given  $\varepsilon > 0$ , consider the set

$$U = \bigcup_{\alpha \in S_m} \bigcap_{k=1}^m \{ \zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi^0)| < \varepsilon \},$$

where  $\alpha = (\alpha_1, \dots, \alpha_m) \in S_m$  denotes the set of permutations of  $\{1, \dots, m\}$  (see Fig. 1 for a diagram of this). Note that  $U$  is, by construction, symmetric, i.e. if

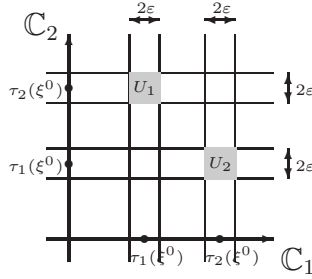


Figure 1:  $U = U_1 \cup U_2$

$(z_1, \dots, z_m) \in U$  then  $(z_{\alpha_1}, \dots, z_{\alpha_m}) \in U$  for all  $(\alpha_1, \dots, \alpha_m) \in S_m$ . Let  $F$  denote the complement to  $U$ :

$$F = \bigcap_{\alpha \in S_m} \{ \zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi^0)| \geq \varepsilon \exists k = 1, \dots, m \}.$$

We need to show that there exists  $\delta > 0$  such that  $(\tau_1(\xi), \dots, \tau_m(\xi)) \in U$  whenever  $|\xi - \xi^0| < \delta$ ; note:

- $\rho^{-1}(\rho(F)) = F$  by construction—if  $\rho(w) = \rho(w')$  then both  $w$  and  $w'$  give rise to the same polynomial, and hence their entries are permutations of each other, and so either both or neither lie in  $F$ ;
- by the surjectivity of  $\rho$ ,

$$\rho(U) = \rho(F^c) = \rho([\rho^{-1}(\rho(F))]^c) = \rho(\rho^{-1}(\rho(F)^c)) = \rho(F)^c;$$

- $\rho(F)$  is closed since  $F$  a closed set and  $\rho$  is a closed mapping;

therefore,  $\rho(U)$  is open. Thus, there exists an open ball in  $\rho(U)$  of radius  $\delta'$  (for some  $\delta' > 0$ ) about  $a(\xi^0) \equiv (a_1(\xi^0), \dots, a_m(\xi^0)) = \rho(\tau_1(\xi^0), \dots, \tau_m(\xi^0))$ :

$$B_{\delta'}(a(\xi^0)) = \{ (c_1, \dots, c_m) \in \mathbb{C}^m : |c_j - a_j(\xi^0)| < \delta' \forall j = 1, \dots, m \} \subset \rho(U).$$

By the continuity of the  $a_j(\xi)$ , there exists  $\delta > 0$  such that

$$|\xi - \xi^0| < \delta \implies |a_j(\xi) - a_j(\xi^0)| < \delta' \text{ for all } j = 1, \dots, m;$$

hence,

$$|\xi - \xi^0| < \delta \implies (a_1(\xi), \dots, a_m(\xi)) \in B_{\delta'}(a(\xi^0)) \subset \rho(U).$$

Finally, since  $\rho(\tau_1(\xi), \dots, \tau_m(\xi)) = (a_1(\xi), \dots, a_m(\xi))$  and  $U$  is symmetric (this is needed as different root orderings give the same coefficients), we find that we have  $(\tau_1(\xi), \dots, \tau_m(\xi)) \in U$  when  $|\xi - \xi^0| < \delta$  as required; this completes the proof of the lemma.  $\square$

Now, let us turn to proving properties of the characteristic roots.

**Proposition 3.4.** *Let  $L = L(D_t, D_x)$  be a linear  $m^{\text{th}}$  order constant coefficient differential operator in  $D_t$  with coefficients that are pseudo-differential operators in  $x$ , with symbol*

$$L(\tau, \xi) = \tau^m + \sum_{j=1}^m P_j(\xi) \tau^{m-j} + \sum_{j=1}^m a_j(\xi) \tau^{m-j},$$

where  $P_j(\lambda\xi) = \lambda^j P_j(\xi)$  for all  $\lambda \gg 1$ ,  $|\xi| \gg 1$ , and  $a_j \in S^{j-\epsilon}$ , for some  $\epsilon > 0$ .

Then each of the characteristic roots of  $L$ , denoted  $\tau_1(\xi), \dots, \tau_m(\xi)$ , is continuous in  $\mathbb{R}^n$ ; furthermore, for each  $k = 1, \dots, m$ , the characteristic root  $\tau_k(\xi)$  is smooth away from multiplicities, and analytic if the operator  $L(D_t, D_x)$  is differential.

If operator  $L(D_t, D_x)$  is strictly hyperbolic, then there exists a constant  $M$  such that, if  $|\xi| \geq M$  then the characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  of  $L$  are pairwise distinct.

*Proof.* The first part of Proposition is simple. Let us now investigate the structure of the characteristic determinant. We use the notation and results from Chapter 12 of [GKZ94] concerning the discriminant  $\Delta_p$  of the polynomial  $p(x) = p_m x^m + \dots + p_1 x + p_0$ ,

$$\Delta_p \equiv \Delta(p_0, \dots, p_m) := (-1)^{\frac{m(m-1)}{2}} p_m^{2m-2} \prod_{i < j} (x_i - x_j)^2,$$

where the  $x_j$  ( $j = 1, \dots, m$ ) are the roots of  $p(x)$ ; that is, the irreducible polynomial in the coefficients of the polynomial which vanishes when the polynomial has multiple roots. We note that  $\Delta_p$  is a continuous function of the coefficients  $p_0, \dots, p_m$  of  $p(x)$  and it is a homogeneous function of degree  $2m - 2$  in them; in addition, it satisfies the quasi-homogeneity property:

$$\Delta(p_0, \lambda p_1, \lambda^2 p_2, \dots, \lambda^m p_m) = \lambda^{m(m-1)} \Delta(p_0, \dots, p_m).$$

Furthermore,  $\Delta_p = 0$  if and only if  $p(x)$  has a double root.

We write  $L(\tau, \xi)$  in the form

$$L(\tau, \xi) = L_m(\tau, \xi) + a_1(\xi) \tau^{m-1} + a_2(\xi) \tau^{m-2} + \dots + a_{m-1}(\xi) \tau + a_m(\xi),$$

where

$$L_m(\tau, \xi) = \tau^m + \sum_{j=1}^m P_j(\xi) \tau^{m-j}$$

is the principal part of  $L(\tau, \xi)$ ; note that the  $P_j(\xi)$  are homogeneous of degree  $j$  and the  $a_j(\xi)$  are symbols of degree  $< j$ . By the homogeneity and quasi-homogeneity properties of  $\Delta_L$ , we have, for  $\lambda \neq 0$ ,

$$\begin{aligned} \Delta_L(\lambda \xi) &= \Delta(P_m(\lambda \xi) + a_m(\lambda \xi), \dots, P_1(\lambda \xi) + a_1(\lambda \xi), 1) \\ &= \Delta(\lambda^m [P_m(\xi) + \frac{a_m(\lambda \xi)}{\lambda^m}], \dots, \lambda [P_1(\xi) + \frac{a_1(\lambda \xi)}{\lambda}], 1) \\ &= \lambda^{m(2m-2)} \Delta(P_m(\xi) + \frac{a_m(\lambda \xi)}{\lambda^m}, \dots, \lambda^{-(m-1)} [P_1(\xi) + \frac{a_1(\lambda \xi)}{\lambda}], \lambda^{-m}) \\ &\quad \text{(using that } \Delta \text{ is homogenous of degree } 2m-2) \\ &= \lambda^{m(m-1)} \Delta(P_m(\xi) + \frac{a_m(\lambda \xi)}{\lambda^m}, \dots, P_1(\xi) + \frac{a_1(\lambda \xi)}{\lambda}, 1) \\ &\quad \text{(by quasi-homogeneity).} \end{aligned}$$

Now, since  $L$  is strictly hyperbolic, the characteristic roots  $\varphi_1(\xi), \dots, \varphi_m(\xi)$  of  $L_m$  are pairwise distinct for  $\xi \neq 0$ , so

$$\Delta_{L_m}(\xi) = \Delta(P_m(\xi), \dots, P_1(\xi), 1) \neq 0 \text{ for } \xi \neq 0.$$

Since the discriminant is continuous in each argument, there exists  $\delta > 0$  such that if  $|\frac{a_j(\lambda \xi)}{\lambda^j}| < \delta$  for all  $j = 1, \dots, m$  then

$$|\Delta(P_m(\xi) + \frac{a_m(\lambda \xi)}{\lambda^m}, \dots, P_1(\xi) + \frac{a_1(\lambda \xi)}{\lambda}, 1)| \neq 0,$$

and hence the roots of the associated polynomial are pairwise distinct. So, fix  $\xi \in \{\xi \in \mathbb{R}^n : |\xi| = 1\}$  and let  $\lambda \rightarrow \infty$ . Since the  $a_j(\xi)$  are polynomials of degree  $< j$  it follows that when  $|\xi| \geq M$ , the characteristic roots of  $L$  are pairwise distinct.  $\square$

### 3.2 Symbolic properties

In this section we will establish a number of useful properties of characteristic roots which will be important for the subsequent analysis. In particular, we will show that asymptotically roots behave like symbols, and we will show the relation between roots of the full symbol of a strictly hyperbolic operator with homogeneous roots of the principal part.

**Proposition 3.5** (Symbolic properties of roots). *Let  $L = L(D_t, D_x)$  be a hyperbolic operator of the following form*

$$L(D_t, D_x) = D_t^m + \sum_{j=1}^m P_j(D_x) D_t^{m-j} + \sum_{j=1}^m \sum_{|\alpha|+m-j=K} c_{\alpha,j}(D_x) D_t^{m-j},$$

where  $P_j(\lambda \xi) = \lambda^j P_j(\xi)$  for  $\lambda \gg 1$ ,  $|\xi| \gg 1$ , and  $c_{\alpha,j} \in S^{|\alpha|}$ . Here  $0 \leq K \leq m-1$  is the maximum order of the lower order terms of  $L$ . Let  $\tau_1(\xi), \dots, \tau_m(\xi)$  denote its characteristic roots; then

I. for each  $k = 1, \dots, m$ , there exists a constant  $C > 0$  such that

$$|\tau_k(\xi)| \leq C(1 + |\xi|) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Furthermore, if we insist that  $L$  is strictly hyperbolic, and denote the roots of the principal part  $L_m(\tau, \xi)$  by  $\varphi_1(\xi), \dots, \varphi_m(\xi)$ , then we have the following properties as well:

II. For each  $\tau_k(\xi)$ ,  $k = 1, \dots, m$ , there exists a corresponding root of the principal symbol  $\varphi_k(\xi)$  (possibly after reordering) such that

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq C(1 + |\xi|)^{K+1-m} \quad \text{for all } \xi \in \mathbb{R}^n. \quad (3.1)$$

In particular, for arbitrary lower terms, we have

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq C \quad \text{for all } \xi \in \mathbb{R}^n. \quad (3.2)$$

III. There exists  $M > 0$  such that, for each characteristic root of  $L$  and for each multi-index  $\alpha$ , we can find constants  $C = C_{k,\alpha} > 0$  such that

$$|\partial_\xi^\alpha \tau_k(\xi)| \leq C|\xi|^{1-|\alpha|} \quad \text{for all } |\xi| \geq M, \quad (3.3)$$

In particular, there exists a constant  $C > 0$  such that

$$|\nabla \tau_k(\xi)| \leq C \quad \text{for all } |\xi| \geq M. \quad (3.4)$$

IV. There exists  $M > 0$  such that, for each  $\tau_k(\xi)$  a corresponding root of the principal symbol  $\varphi_k(\xi)$  can be found (possibly after reordering) which satisfies, for each multi-index  $\alpha$  and  $k = 1, \dots, m$ ,

$$|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)| \leq C|\xi|^{K+1-m-|\alpha|} \quad \text{for all } |\xi| \geq M \quad (3.5)$$

In particular, since  $K \leq m - 1$ , we have

$$|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)| \leq C|\xi|^{-|\alpha|} \quad \text{for all } |\xi| \geq M, \quad (3.6)$$

for each multi-index  $\alpha$  and  $k = 1, \dots, m$ .

First, we need the following lemma about perturbation properties of general smooth functions. Clearly, we do not need to require that functions are smooth, but this will be the case in our application.

**Lemma 3.6.** *Let  $p, q : \mathbb{C} \rightarrow \mathbb{C}$  be smooth functions and suppose  $z_0$  is a simple zero of  $p(z)$  (i.e.  $p(z_0) = 0$ ,  $p'(z_0) \neq 0$ ). Consider, for each  $\varepsilon > 0$ , the following “perturbation” of  $p(z)$ :*

$$p_\varepsilon(z) := p(z) + \varepsilon q(z),$$

*and suppose  $z_\varepsilon$  is a root of  $p_\varepsilon(z)$ ; then, for all sufficiently small  $\varepsilon > 0$ , we have*

$$|z_\varepsilon - z_0| \leq C\varepsilon \left| \frac{q(z_0)}{p'(z_0)} \right|. \quad (3.7)$$

*Proof.* By Taylor's theorem, we have, near  $z_0$ ,

$$\begin{aligned} p_\varepsilon(z) &= p_\varepsilon(z_0) + p'_\varepsilon(z_0)(z - z_0) + O(|z - z_0|^2) \\ &= \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(z - z_0) + O(|z - z_0|^2). \end{aligned}$$

Thus, setting  $z = z_\varepsilon$ , we get

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(z_\varepsilon - z_0) + O(|z_\varepsilon - z_0|^2). \quad (3.8)$$

Now, consider the function of  $\varepsilon$ ,  $z(\varepsilon) := z_\varepsilon$ ; this is clearly smooth since  $p$  and  $q$  are smooth and  $z_0$  is a simple zero of  $p(z)$ . Indeed,  $p'_\varepsilon(z_\varepsilon) \approx p'(z_0) \neq 0$  for small  $\varepsilon$ , hence  $z_\varepsilon$  is a simple root of  $p_\varepsilon$ . Thus, near the origin,

$$z(\varepsilon) = z(0) + \varepsilon z'(0) + O(\varepsilon^2). \quad (3.9)$$

Combining (3.8) and (3.9), we get

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(\varepsilon z'(0) + O(\varepsilon^2)) + O(\varepsilon^2),$$

or,

$$0 = q(z_0) + p'(z_0)z'(0) + O(\varepsilon),$$

for small  $\varepsilon$ . Therefore, by the triangle inequality, for each  $\varepsilon > 0$  small enough,

$$|z'(0)| \leq \frac{C\varepsilon}{|p'(z_0)|} + \left| \frac{q(z_0)}{p'(z_0)} \right|,$$

and, thus,

$$|z'(0)| \leq C \left| \frac{q(z_0)}{p'(z_0)} \right|. \quad (3.10)$$

Finally, combining (3.10) with (3.9), we obtain (3.7) as required.  $\square$

*Proof of Proposition 3.5.*

**Part I:** We may write  $L(\tau, \xi)$  in the form

$$L(\tau, \xi) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_{m-1}(\xi)\tau + a_m(\xi),$$

where  $|a_j(\xi)| \leq C\langle \xi \rangle^j$ . Hence for all  $k$  we have  $|\tau_k(\xi)| \leq C\langle \xi \rangle$  by Lemma 3.1.

**Part II:** In the proof of this part, let us write  $L(\tau, \xi)$  in the form

$$L(\tau, \xi) = \sum_{i=0}^R L_{m-r_i}(\tau, \xi),$$

where  $r_0 = 0$ ,  $m - r_1 = K$  (the maximum order of the lower order terms),  $1 \leq r_1 < \dots < r_R \leq m$ ,

$$L_m(\tau, \xi) = \tau^m + \sum_{j=1}^m P_j(\xi) \tau^{m-j}$$

$$\text{and } L_{m-r_i}(\tau, \xi) = \sum_{|\alpha|+j=m-r_i} c_{\alpha,j}(\xi) \tau^j \text{ for } 1 \leq i \leq R;$$

here, as usual, the  $P_j(\xi)$  are homogeneous in  $\xi$  of order  $j$ .

Denote the roots of

$$\mathbb{L}_l(\tau, \xi) := \sum_{i=0}^l L_{m-r_i}(\tau, \xi), \quad 0 \leq l \leq R,$$

with respect to  $\tau$  by  $\tau_1^l(\xi), \dots, \tau_m^l(\xi)$ . Note that  $\mathbb{L}_0(\tau, \xi) = L_m(\tau, \xi)$ , i.e.  $\mathbb{L}_0(\tau, \xi)$  is the principal symbol with no lower order terms. Since  $\mathbb{L}_l(\tau, \xi)$  are strictly hyperbolic, we will look at  $|\xi| \geq M_0$ , where all  $\tau_1^l(\xi), \dots, \tau_m^l(\xi)$  are distinct, for all  $l$ .

We shall show that there exists  $M \geq M_0$  so that, possibly after reordering the roots, for all  $k = 1, \dots, m$ ,

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \leq C|\xi|^{-r_{l+1}+1} \text{ for all } l = 0, \dots, R-1 \text{ and } |\xi| \geq M. \quad (3.11)$$

Assuming this, and noting that  $\tau_k^0(\xi) = \varphi_k(\xi)$  and  $\tau_k^R(\xi) = \tau_k(\xi)$  for each  $k = 1, \dots, m$  (possibly after reordering), we obtain

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq \sum_{l=0}^{R-1} |\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \leq C|\xi|^{-r_1+1} \text{ when } |\xi| \geq M;$$

this, together with the continuity of the  $\tau_k(\xi)$  and  $\varphi_k(\xi)$ —and thus the boundedness of  $|\tau_k(\xi) - \varphi_k(\xi)|$  in  $B_M(0)$ , gives (3.1). Then, (3.2) follows by setting  $K = m - 1$ . Here we also used  $r_1 = m - K$ .

So, with the aim of proving (3.11), we first introduce some notation: set

$$\begin{aligned} \tilde{L}_{m-r_i} : \mathbb{C} \times \mathbb{S}^{n-1} &\rightarrow \mathbb{C} : \quad \tilde{L}_{m-r_i}(\tau, \omega) = L_{m-r_i}(\tau, \omega), \quad i = 0, \dots, R, \\ \tilde{\mathbb{L}}_l : (M_0, \infty) \times \mathbb{C} \times \mathbb{S}^{n-1} &\rightarrow \mathbb{C} : \quad \tilde{\mathbb{L}}_l(\rho, \tau, \omega) = \rho^{-m} \mathbb{L}_l(\rho\tau, \rho\omega), \quad l = 0, \dots, R; \end{aligned}$$

observe that  $\tilde{L}_{m-r_i}$  is just the restriction of  $L_{m-r_i}(\tau, \xi)$  to  $\mathbb{C} \times \mathbb{S}^{n-1}$ . Denote by  $\tilde{\varphi}_1(\omega), \tilde{\varphi}_2(\omega), \dots, \tilde{\varphi}_m(\omega)$  the roots of  $\tilde{L}_m(\tau, \omega) = \tilde{\mathbb{L}}_0(\rho, \tau, \omega)$  with respect to  $\tau$ , and by  $\tilde{\tau}_1^k(\rho, \omega), \tilde{\tau}_2^k(\rho, \omega), \dots, \tilde{\tau}_m^k(\rho, \omega)$  those of  $\tilde{\mathbb{L}}_k(\rho, \tau, \omega)$ .

We denote  $\tilde{\tau} = \frac{\tau}{|\xi|}$ . Since,

$$\tilde{L}_m(\tilde{\tau}, \frac{\xi}{|\xi|}) = L_m(\tilde{\tau}, \frac{\xi}{|\xi|}) = |\xi|^{-m} L_m(\tau, \xi) = |\xi|^{-m} \mathbb{L}_0(\tau, \xi) = \tilde{\mathbb{L}}_0(|\xi|, \tilde{\tau}, \frac{\xi}{|\xi|})$$



for  $\xi \in \mathbb{R}^n$ ,  $\tau \in \mathbb{C}$ , and

$$\begin{aligned}
\tilde{\mathbb{L}}_{l+1}(\rho, \tau, \omega) &= \rho^{-m} \mathbb{L}_{l+1}(\rho\tau, \rho\omega) = \rho^{-m} \sum_{i=0}^{l+1} L_{m-r_i}(\rho\tau, \rho\omega) \\
&= \rho^{-m} \sum_{i=0}^l L_{m-r_i}(\rho\tau, \rho\omega) + \rho^{-m} \sum_{|\alpha|+j=m-r_{l+1}} c_{\alpha,j}(\rho\omega) (\rho\tau)^j \\
&= \tilde{\mathbb{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} \sum_{|\alpha|+j=m-r_{l+1}} \frac{c_{\alpha,j}(\rho\omega)}{\rho^{|\alpha|}} \tau^j \\
&= \tilde{\mathbb{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} L_{m-r_{l+1}}^0(\rho, \tau, \omega)
\end{aligned}$$

for  $\omega \in S^{n-1}$ ,  $\rho > M_0$ ,  $\tau \in \mathbb{C}$ ,  $l = 0, \dots, R-1$ . Here

$$L_{m-r_{l+1}}^0(\rho, \tau, \omega) = \sum_{|\alpha|+j=m-r_{l+1}} \frac{c_{\alpha,j}(\rho\omega)}{\rho^{|\alpha|}} \tau^j.$$

We also have

$$|\xi|^{-m} \mathbb{L}_L(\tau, \xi) = \tilde{\mathbb{L}}_l(|\xi|, \frac{\xi}{|\xi|}, \tilde{\tau}).$$

As the left-hand side of this is zero when  $\tau = \tau_k^l(\xi)$ ,  $k = 1, \dots, m$ , and the right-hand side is zero when  $\tilde{\tau} = \tilde{\tau}_k^l(|\xi|, \frac{\xi}{|\xi|})$ ,  $k = 1, \dots, m$ , we see that  $|\xi| \tilde{\tau}_k^l(|\xi|, \frac{\xi}{|\xi|}) = \tau_k^l(\xi)$  for each  $k = 1, \dots, m$  (possibly after reordering). Hence, for all  $|\xi| \geq M_0$ ,  $k = 1, \dots, m$  and  $l = 0, \dots, R-1$ , we have

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| = |\tilde{\tau}_k^{l+1}(|\xi|, \frac{\xi}{|\xi|}) - \tilde{\tau}_k^l(|\xi|, \frac{\xi}{|\xi|})| |\xi|.$$

Next, observe that applying Lemma 3.6 with  $\varepsilon = \rho^{-r_{l+1}}$  to

$$\tilde{\mathbb{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} L_{m-r_{l+1}}^0(\rho, \tau, \omega)$$

yields, for all  $\omega \in \mathbb{S}^{n-1}$  and  $k = 1, \dots, m$ ,

$$|\tilde{\tau}_k^{l+1}(\rho, \omega) - \tilde{\tau}_k^l(\rho, \omega)| \leq C \rho^{-r_{l+1}} \left| \frac{L_{m-r_{l+1}}^0(\rho, \tilde{\tau}_k^l(\rho, \omega), \omega)}{\partial_\tau \tilde{\mathbb{L}}_l(\rho, \tilde{\tau}_k^l(\rho, \omega), \omega)} \right|.$$

provided we take  $\rho \geq M'$  for a sufficiently large constant  $M' \geq M_0$ . Therefore, for all  $|\xi| \geq M'$ ,  $k = 1, \dots, m$  and  $l = 0, \dots, R-1$ , we have

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \leq C |\xi|^{-r_{l+1}+1} \left| \frac{L_{m-r_{l+1}}^0(|\xi|, \frac{\tau_k^l(\xi)}{|\xi|}, \frac{\xi}{|\xi|})}{\partial_\tau \tilde{\mathbb{L}}_l(|\xi|, \frac{\tau_k^l(\xi)}{|\xi|}, \frac{\xi}{|\xi|})} \right|. \quad (3.12)$$

Thus, it suffices to show the following two inequalities when  $|\xi| \geq M$  for some  $M \geq M'$ :

- there exists a constant  $C_1$  so that, for all  $1 \leq i \leq R$ ,

$$\left| L_{m-r_i}^0(|\xi|, \frac{\tau_k^l(\xi)}{|\xi|}, \frac{\xi}{|\xi|}) \right| = \left| \sum_{|\alpha|+j=m-r_i} \frac{c_{\alpha,j}(\xi)}{|\xi|^{|\alpha|}} \left( \frac{\tau_k^l(\xi)}{|\xi|} \right)^j \right| \leq C_1; \quad (3.13)$$

and

- there exists a constant  $C_2 > 0$  so that, for all  $0 \leq l \leq R-1$ ,

$$|\partial_\tau \tilde{L}_l(|\xi|, \frac{\tau_k^l(\xi)}{|\xi|}, \frac{\xi}{|\xi|})| = |\xi|^{-m+1} |\partial_\tau L_l(\tau_k^l(\xi), \xi)| \geq C_2. \quad (3.14)$$

Then, combining (3.12), (3.13) and (3.14) gives (3.11).

The first estimate (3.13) follows immediately from Part I since the  $\tau_k^l(\xi)$  are roots of strictly hyperbolic equations, and from the fact that  $c_{\alpha,j} \in S^{|\alpha|}$ .

The second, (3.14), in the case  $l = 0$  is clear: the homogeneity of  $L_m(\tau, \xi)$  and its roots give

$$|\xi|^{-m+1} |\partial_\tau L_0(\tau_k^0(\xi), \xi)| = \left| \partial_\tau L_m(\varphi_k(\frac{\xi}{|\xi|}), \frac{\xi}{|\xi|}) \right|,$$

which is never zero due to the strict hyperbolicity of  $L_m$  and hence (using that the sphere  $S^{n-1}$  is compact and  $L_m(\tau, \xi)$  is continuous and thus achieves its minimum) is bounded below by some positive constant as required.

For  $1 \leq l \leq R-1$ , we know that  $\tau_k^l(\xi)$ ,  $k = 1, \dots, m$ , are simple zeros of  $L_l(\tau, \xi)$  for  $|\xi| \geq M_0$  by the earlier choice of  $M_0$ . Observe,

$$\frac{(\partial_\tau L_l)(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} = \frac{(\partial_\tau L_m)(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} + \sum_{i=1}^l \frac{(\partial_\tau L_{m-r_i})(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}}.$$

Now,

$$\frac{(\partial_\tau L_{m-r_i})(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} \rightarrow 0 \text{ as } |\xi| \rightarrow \infty$$

for  $i = 1, \dots, l$ , because  $\partial_\tau L_{m-r_i}(\tau, \xi)$  is a symbol of order  $m - r_i - 1$ . Also, using the Mean Value Theorem,

$$\begin{aligned} (\partial_\tau L_m)(\tau_k^l(\xi), \xi) &= (\partial_\tau L_m)(\varphi_k(\xi), \xi) + [(\partial_\tau L_m)(\tau_k^l(\xi), \xi) - (\partial_\tau L_m)(\varphi_k(\xi), \xi)] \\ &= (\partial_\tau L_m)(\varphi_k(\xi), \xi) + (\partial_\tau^2 L_m)(\bar{\tau}_k^l(\xi), \xi), \end{aligned}$$

where  $\bar{\tau}_k^l(\xi)$  lies on the line connecting  $\varphi_k(\xi)$  and  $\tau_k^l(\xi)$  for each  $\xi \in \mathbb{R}^n$ ,  $k = 1, \dots, m$  and  $l = 1, \dots, R-1$ , and

$$\frac{|(\partial_\tau^2 L_m)(\bar{\tau}_k^l(\xi), \xi)|}{|\xi|^{m-1}} \leq C|\xi|^{-1} \rightarrow 0 \text{ as } |\xi| \rightarrow \infty.$$

Therefore, for a sufficiently large constant  $M \geq M'$ , there exists a constant  $C_2 > 0$  such that

$$\frac{|\partial_\tau L_m(\tau_k^l(\xi), \xi)|}{|\xi|^{m-1}} \geq C \frac{|\partial_\tau L_m(\varphi_k(\xi), \xi)|}{|\xi|^{m-1}} \geq C_2, \text{ when } |\xi| \geq M.$$

This completes the proof of (3.13) and thus of Part II.

**Part III:** We take  $M > 0$  so that for  $|\xi| \geq M$ , the roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  are distinct.

To prove the statement, we do induction on  $|\alpha|$ .

First, assume  $|\alpha| = 1$ . Since  $L(\tau_k(\xi), \xi) = 0$  for each  $k = 1, \dots, m$ , we have, for each  $i = 1, \dots, n$ ,

$$\frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi) + \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \frac{\partial \tau_k}{\partial \xi_i}(\xi) = 0.$$

The first term is a symbol of order  $m-1$  in  $(\tau_k(\xi), \xi)$ , hence, by Part I, there exists a constant  $C$  such that, when  $|\xi| \geq M_1$  for some suitably large constant  $M_1 \geq M$ ,

$$\left| \frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi) \right| \leq C|\xi|^{m-1}.$$

The inequality (3.3) for  $|\alpha| = 1$  (i.e. (3.4)) then follows immediately from:

**Lemma 3.7.** *There exists constants  $C > 0$ ,  $M_2 \geq M$  such that, for each  $k = 1, \dots, m$ ,*

$$\left| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \right| \geq C|\xi|^{m-1} \quad \text{when } |\xi| \geq M_2.$$

*Proof.* Note that

$$\left| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \right| \geq \left| \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \right| - \left| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \right|, \quad (3.15)$$

where  $L_m(\tau, \xi)$  is the principal symbol of  $L$  and  $\varphi_1(\xi), \dots, \varphi_m(\xi)$  are the corresponding characteristic roots, ordered in the same way as in Part II. We look at each of the terms on the right-hand side in turn:

- By strict hyperbolicity,  $\frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)$  is non-zero for  $\xi \neq 0$ . Thus, for all  $\xi \neq 0$ ,

$$\left| \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \right| = |\xi|^{m-1} \left| \frac{\partial L_m}{\partial \tau} \left( \frac{\xi}{|\xi|}, \varphi \left( \frac{\xi}{|\xi|} \right) \right) \right| \geq C|\xi|^{m-1}. \quad (3.16)$$

- Observe,

$$\begin{aligned} & \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \\ &= \frac{\partial L_m}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) + \sum_{r=0}^{m-1} \sum_{|\alpha|+l=r} l c_{\alpha,l}(\xi) \tau_k(\xi)^{l-1}. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{\partial L_m}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \\ &= m(\tau_k(\xi)^{m-1} - \varphi_k(\xi)^{m-1}) + \sum_{j=1}^m (m-j) P_j(\xi) (\tau_k(\xi)^{m-j-1} - \varphi_k(\xi)^{m-j-1}), \end{aligned}$$

and

$$|\tau_k(\xi)^r - \varphi_k(\xi)^r| = |\tau_k(\xi) - \varphi_k(\xi)| |\tau_k(\xi)^{r-1} + \tau_k(\xi)^{r-2}\varphi_k(\xi) + \cdots + \varphi_k(\xi)^{r-1}|.$$

So, by Part I and Part II (specifically inequality (3.2)) and the fact that the  $P_j(\xi)$  are homogeneous in  $\xi$  of order  $j$ , we have, for some suitably large  $M_2 \geq M$ ,

$$\left| \frac{\partial L_m}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \right| \leq C|\xi|^{m-2} \quad \text{when } |\xi| \geq M_2.$$

This, together with

$$\left| \sum_{|\alpha|+l=r} l c_{\alpha,r}(\xi) \tau_k(\xi)^{l-1} \right| \leq C|\xi|^{r-1} \leq C|\xi|^{m-2} \quad \text{when } |\xi| \geq M_2, \quad r = 0, \dots, m-1,$$

which again follows straight from Part I, yields

$$\left| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \right| \leq C|\xi|^{m-2} \quad \text{for } |\xi| \geq M_2. \quad (3.17)$$

The result now follows by combining (3.15), (3.17) and (3.16). The proof of Lemma 3.7 is complete.  $\square$

For  $|\alpha| = J > 1$ , assume inductively that,

$$|\partial_\xi^\alpha \tau_k(\xi)| \leq C|\xi|^{1-|\alpha|} \quad \text{when } |\xi| \geq M, \quad |\alpha| \leq J-1,$$

for some fixed  $M \geq \max(M_1, M_2)$ .

Then, for  $|\alpha| = J$ , we use  $\partial_\xi^\alpha [L(\tau_k(\xi), \xi)] = 0$ , i.e.

$$\begin{aligned} & \partial_\xi^\alpha \tau_k(\xi) \partial_\tau L(\tau_k(\xi), \xi) \\ & + \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \tau_k(\xi) \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L(\tau_k(\xi), \xi) = 0. \end{aligned}$$

By the inductive hypothesis and the fact that  $\partial_\xi^\beta \partial_\tau^j L(\tau_k(\xi), \xi)$  is a symbol of order  $m-j-|\beta|$ , we have, for all multi-indices  $\beta^1, \dots, \beta^r \neq 0$  or  $\alpha$  satisfying  $\beta^1 + \dots + \beta^r \leq \alpha$ ,

$$\left| \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \tau_k(\xi) \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L(\tau_k(\xi), \xi) \right| \leq C_{k,\alpha} |\xi|^{m-|\alpha|} \quad \text{when } |\xi| \geq M.$$

Thus, using Lemma 3.7 again, we have

$$|\partial_\xi^\alpha \tau_k(\xi)| \leq \frac{C_\alpha |\xi|^{m-|\alpha|}}{|\partial_\tau L(\tau_k(\xi), \xi)|} \leq C_{k,\alpha} |\xi|^{1-|\alpha|} \quad \text{when } |\xi| \geq M,$$

which completes the proof of the induction step.

**Part IV:** Once again, assume that the roots  $\tau_k(\xi)$ ,  $k = 1, \dots, m$ , correspond to  $\varphi_k(\xi)$ ,  $k = 1, \dots, m$ , in the manner of Part II.

The proof of this part for general multi-index  $\alpha$  is quite technical, so we first give the proof in the case  $|\alpha| = 1$  to demonstrate the main ideas required, and then show how it can be extended when  $|\alpha| > 1$ .

From  $L(\tau_k(\xi), \xi) = 0 = L_m(\varphi_k(\xi), \xi)$ , we have for each  $i = 1, \dots, n$ ,

$$\begin{aligned} \frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi) + \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \frac{\partial \tau_k}{\partial \xi_i}(\xi) &= 0, \\ \frac{\partial L_m}{\partial \xi_i}(\varphi_k(\xi), \xi) + \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \frac{\partial \varphi_k}{\partial \xi_i}(\xi) &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \left( \frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi) \right) &= \frac{\partial L_m}{\partial \xi_i}(\varphi_k(\xi), \xi) - \frac{\partial L_m}{\partial \xi_i}(\tau_k(\xi), \xi) \\ &+ \frac{\partial \varphi_k}{\partial \xi_i} \left[ \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) - \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \right] - \frac{\partial(L - L_m)}{\partial \xi_i}(\tau_k(\xi), \xi). \end{aligned} \quad (3.18)$$

It suffices to show that the right-hand side is bounded absolutely by  $C|\xi|^{m-2}$  when  $|\xi| \geq M_1$  for some suitably large  $M_1 \geq M_0$ ; this is because an application of Lemma 3.7 then yields

$$\left| \frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi) \right| \leq \frac{C|\xi|^{m-2}}{\left| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \right|} \leq C|\xi|^{-1} \quad \text{for } |\xi| \geq M,$$

where  $M = \max(M_1, M_2)$ .

Since  $\partial_{\xi_i}(L - L_m)(\tau, \xi)$  is a symbol of order  $\leq m-2$  in  $(\tau, \xi)$ , it is immediately clear that the final term of (3.18) is bounded by  $C|\xi|^{m-2}$ ; here we have also used Part I. Also, noting that  $|\partial_{\xi_i} \varphi_k(\xi)| \leq C$  by the homogeneity of  $\varphi_k(\xi)$ , we have, by (3.17),

$$\left| \frac{\partial \varphi_k}{\partial \xi_i}(\xi) \right| \left| \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\tau_k(\xi), \xi) \right| \leq C|\xi|^{m-2}.$$

Finally, by the Mean Value Theorem,

$$\left| \frac{\partial L_m}{\partial \xi_i}(\varphi_k(\xi), \xi) - \frac{\partial L_m}{\partial \xi_i}(\tau_k(\xi), \xi) \right| \leq C \left| \frac{\partial^2 L_m}{\partial \tau \partial \xi_i}(\xi, \bar{\tau}) \right| |\varphi_k(\xi) - \tau_k(\xi)|,$$

where  $\bar{\tau}$  lies on the linear path between  $\varphi_k(\xi)$  and  $\tau_k(\xi)$ —which means that (using Part I once more)  $|\bar{\tau}| \leq C|\xi|$  for  $|\xi| \geq M$ . Since  $\partial_{\tau} \partial_{\xi_i} L_m(\tau, \xi)$  is a symbol of order  $m-2$  in  $(\tau, \xi)$ , and  $|\varphi_k(\xi) - \tau_k(\xi)| \leq C$  by Part II, this term is bounded by  $C|\xi|^{m-2}$ , completing the proof in the case  $|\alpha| = 1$ .

For  $|\alpha| = J > 1$ , we assume inductively that

$$|\partial_{\xi}^{\alpha} \tau_k(\xi) - \partial_{\xi}^{\alpha} \varphi_k(\xi)| \leq C|\xi|^{-|\alpha|} \quad \text{for } |\xi| \geq M, |\alpha| \leq J-1.$$

As in the proof of Part III, we have

$$\begin{aligned} & \partial_\xi^\alpha \tau_k(\xi) \partial_\tau L(\tau_k(\xi), \xi) \\ & + \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \tau_k(\xi) \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L(\tau_k(\xi), \xi) = 0; \end{aligned}$$

similarly,

$$\begin{aligned} & \partial_\xi^\alpha \varphi_k(\xi) \partial_\tau L_m(\varphi_k(\xi), \xi) \\ & + \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \varphi_k(\xi) \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\varphi_k(\xi), \xi) = 0. \end{aligned}$$

Thus,

$$\begin{aligned} & (\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)) \partial_\tau L(\tau_k(\xi), \xi) = \\ & \quad \partial_\xi^\alpha \varphi_k(\xi) (\partial_\tau L_m(\varphi_k(\xi), \xi) - \partial_\tau L(\tau_k(\xi), \xi)) \\ & + \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \varphi_k(\xi) \right) [\partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\varphi_k(\xi), \xi) - \\ & \quad \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\tau_k(\xi), \xi)] \\ & + \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r [\partial_\xi^{\beta^j} \varphi_k(\xi) - \partial_\xi^{\beta^j} \tau_k(\xi)] \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\tau_k(\xi), \xi) \\ & - \sum_{\substack{\beta^1 + \dots + \beta^r \leq \alpha, \\ \beta^j \neq 0, \beta^j \neq \alpha}} c_{\alpha, \beta^1, \dots, \beta^r} \left( \prod_{j=1}^r \partial_\xi^{\beta^j} \tau_k(\xi) \right) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r (L - L_m)(\tau_k(\xi), \xi). \end{aligned}$$

We claim the right-hand side is then bounded absolutely by  $C_\alpha |\xi|^{m-1-|\alpha|}$ , which, together with Lemma 3.7, yields the desired estimate.

To see this, let us look at each of the terms in turn:

- $|\partial_\xi^\alpha \varphi_k(\xi)| \leq C_\alpha |\xi|^{1-|\alpha|}$  by the homogeneity of  $\varphi_k(\xi)$ ; using this with (3.17) gives the desired bound.
- Using the Mean Value Theorem as in the case  $|\alpha| = 1$ , we get

$$\begin{aligned} & |[\partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\varphi_k(\xi), \xi) - \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L_m(\tau_k(\xi), \xi)]| \\ & \leq C_\alpha |\xi|^{m-|\alpha|+|\beta^1|+\dots+|\beta^r|-r-1}; \end{aligned}$$

coupled with  $|\partial_\xi^\beta \varphi_k(\xi)| \leq C_\alpha |\xi|^{1-|\beta|}$ , this gives the correct bound.

- By the inductive hypothesis,

$$|\partial_\xi^{\beta^j} \varphi_k(\xi) - \partial_\xi^{\beta^j} \tau_k(\xi)| \leq C_\beta |\xi|^{1-|\beta^j|};$$

together with

$$|\partial_\xi^{\alpha-\beta^1-\dots-\beta^r} \partial_\tau^r L_m(\tau_k(\xi), \xi)| \leq C_\alpha |\xi|^{m-|\alpha|+|\beta^1|+\dots+|\beta^r|-r},$$

which follows from Part I and the homogeneity of  $L_m(\tau, \xi)$ , this gives the correct estimate.

- To show the final term is bounded absolutely by  $|\xi|^{m-1-|\alpha|}$ , first note that

$$\partial_\xi^{\alpha-\beta^1-\dots-\beta^r} \partial_\tau^r (L - L_m)(\tau_k(\xi), \xi)$$

is a symbol of order  $\leq m - |\alpha| + |\beta^1| + \dots + |\beta^r| - r - 1$ ; applying Part III to estimate the  $\partial_\xi^{\beta^j} \tau_k(\xi)$  terms, we have the required result.

This completes the proof of (3.6); (3.5) is proved in a similar way in the proof using the set-up of the proof of Part II. The proof of Proposition 3.5 is now complete.  $\square$

We will now establish further symbolic properties of characteristic roots. A refinement of this proposition concerning real and imaginary parts of complex roots  $\tau$  is given in Proposition 6.16.

**Proposition 3.8.** *Suppose that the characteristic roots  $\phi_k$ ,  $k = 1, \dots, m$ , of the principal part  $L_m(\tau, \xi)$  of a strictly hyperbolic operator  $L(\tau, \xi)$  in (2.1) are non-zero for all  $\xi \neq 0$ . Then the roots  $\tau(\xi)$  of the full symbols satisfy the following properties:*

- (i) *for all multi-indices  $\alpha$  there exists a constants  $M, C_\alpha > 0$  such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha |\xi|^{1-|\alpha|};$$

*for all  $|\xi| \geq M$ .*

- (ii) *there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\tau(\xi)| \geq C|\xi|$ ;*
- (iii) *there exists a constant  $C_0 > 0$  such that  $|\partial_\omega \tau(\lambda\omega)| \geq C_0$  for all  $\omega \in \mathbb{S}^{n-1}$ ,  $\lambda > 0$ ; in particular,  $|\nabla \tau(\xi)| \geq C_0$  for all  $\xi \in \mathbb{R}^n \setminus \{0\}$ ;*
- (iv) *there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,*

$$\frac{1}{\lambda} \Sigma_\lambda(\tau) \equiv \frac{1}{\lambda} \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0).$$

*Proof.* • Property (i): by Proposition 3.5, Part III,

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha |\xi|^{1-|\alpha|} \quad \text{for all } |\xi| \geq M,$$

for all multi-indices  $\alpha$ .



- Properties (ii) and (iii): these follow by using perturbation methods. By Proposition 3.5, Part IV, there exists a homogeneous function  $\varphi(\xi)$  of order 1 such that, for all  $|\xi| \geq M$  and  $k = 1, \dots, n$ ,

$$|\tau(\xi) - \varphi(\xi)| \leq C_0 \text{ and } |\partial_{\xi_k} \tau(\xi) - \partial_{\xi_k} \varphi(\xi)| \leq C_k |\xi|^{-1},$$

for some constants  $C_0, C_k > 0$ . Now, the homogeneity of  $\varphi(\xi)$  implies that  $\varphi(\xi) = |\xi| \varphi(\frac{\xi}{|\xi|})$  and  $e_k \cdot \nabla \varphi(e_k) = \varphi(e_k)$ , where  $e_k = (\underbrace{0, \dots, 0}_k, 1, 0, \dots, 0)$ , so

$$|\varphi(\xi)| \geq C' |\xi| \text{ for all } \xi \in \mathbb{R}^n \text{ and } |\partial_\omega \varphi(\lambda \omega)| \geq C' \text{ for all } \omega \in \mathbb{S}^{n-1}, \lambda > 0,$$

for some constant  $C' > 0$ . Thus,

$$|\tau(\xi)| \geq |\varphi(\xi)| - |\tau(\xi) - \varphi(\xi)| \geq C' |\xi| - C_0 \geq C |\xi| \text{ for } |\xi| \geq M, \quad (3.19)$$

for some constants  $M, C > 0$ , and

$$|\partial_\omega \tau(\lambda \omega)| \geq |\partial_\omega \varphi(\lambda \omega)| - |\partial_\omega \varphi(\lambda \omega) - \partial_\omega \tau(\lambda \omega)| \geq C' - C_k \lambda^{-1} \geq C > 0$$

for all  $\omega \in \mathbb{S}^{n-1}$  and suitably large  $\lambda$ ; for small  $\lambda > 0$ ,  $\partial_\omega \tau(\lambda \omega)$  is separated from 0 by the convexity condition, so  $|\partial_\omega \tau(\lambda \omega)| \geq C > 0$  for all  $\omega \in \mathbb{S}^{n-1}$ ,  $\lambda > 0$ , as required.

- Property (iv)—there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,  $\frac{1}{\lambda} \Sigma_\lambda(\tau) \subset B_{R_1}(0)$ —holds by Proposition 3.5, Part II, and the fact that  $\frac{1}{\lambda} \Sigma_\lambda(\varphi) = \Sigma_1(\varphi)$  for the characteristic root of the principal symbol  $\varphi$  corresponding to  $\tau$ .

□

## 4 Oscillatory integrals with convexity

As discussed in Section 1.2, in the case of homogeneous  $m^{\text{th}}$  order strictly hyperbolic operators, geometric properties of the characteristic roots play the fundamental role in determining the  $L^p - L^q$  decay; in particular, if the characteristic roots satisfy the convexity condition of Definition 1.1, then the decay is, in general, more rapid than when they do not. We will show that a similar improvement can be obtained for operators with lower order terms when a suitable ‘convexity condition’ holds. In Section 4.3, we shall extend this notion of the convexity condition to functions  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  and prove a decay estimate for an oscillatory integral (related to the solution representation for a strictly hyperbolic operator) with phase function  $\tau$ .

First, we give a general result for oscillatory integrals and show how the concept of functions of “convex type” allows its application to derive the time decay.

## 4.1 Estimates for oscillatory integrals

The following theorem is central in proving results involving convexity conditions. In some sense, it bridges the gap between the man der Corput Lemma and the method of stationary phase, in that the former is used when there is no convexity but gives a weaker result, while the latter can be used when a stronger condition than simply convexity holds and gives a better result. Here, we state and prove a result that has no reference to convexity; however, in the following section, we show how convexity (in some sense) enables this result to be used in applications. An earlier version of this result has appeared in [Ruzh07], with applications to equations with time dependent homogeneous symbols in [MR07]. For completeness we also include a more detailed proof here.

**Theorem 4.1.** *Consider the oscillatory integral*

$$I(\lambda, \nu) = \int_{\mathbb{R}^N} e^{i\lambda\Phi(y, \nu)} A(y, \nu) g(y) dy, \quad (4.1)$$

where  $N \in \mathbb{N}$ ,  $I : [0, \infty) \times \mathcal{N} \rightarrow \mathbb{C}$ ,  $\mathcal{N}$  is any set of parameters  $\nu$ , and

- (I1) *there exists a bounded open set  $U \subset \mathbb{R}^N$  such that  $g \in C_0^\infty(U)$ ;*
- (I2)  *$\Phi(y, \nu)$  is a complex-valued function such that  $\text{Im } \Phi(y, \nu) \geq 0$  for all  $y \in U$ ,  $\nu \in \mathcal{N}$ ;*
- (I3) *for some fixed  $z \in \mathbb{R}^N$ , some  $\delta > 0$ , and some  $\gamma \in \mathbb{N}$ ,  $\gamma \geq 2$ , the function*

$$F(\rho, \omega, \nu) := \Phi(\rho\omega + z, \nu)$$

*satisfies*

$$|\partial_\rho F(\rho, \omega, \nu)| \geq C\rho^{\gamma-1} \text{ and } |\partial_\rho^m F(\rho, \omega, \nu)| \leq C_m \rho^{1-m} |\partial_\rho F(\rho, \omega, \nu)|$$

*for all  $(\omega, \nu) \in \mathbb{S}^{N-1} \times \mathcal{N}$ , all integers  $1 \leq m \leq [N/\gamma] + 1$  and all  $\rho > 0$ , for which  $\rho\omega + z \in U$ ;*

- (I4) *for each multi-index  $\alpha$  such that  $|\alpha| \leq [\frac{N}{\gamma}] + 1$ , there exists a constant  $C_\alpha > 0$  such that  $|\partial_y^\alpha A(y, \nu)| \leq C_\alpha$  for all  $y \in U$ ,  $\nu \in \mathcal{N}$ .*

*Then there exists a constant  $C = C_{N, \gamma} > 0$  such that*

$$|I(\lambda, \nu)| \leq C(1 + \lambda)^{-\frac{N}{\gamma}} \text{ for all } \lambda \in [0, \infty), \nu \in \mathcal{N}. \quad (4.2)$$

Constant  $C$  in (4.2) is independent of  $\lambda$  and  $\nu$ .

**Remark 4.2.** *This theorem extends to the case where  $A(y, \nu)$  is replaced by  $A(y, \nu')$ , where  $\nu'$  may be independent of the variable  $\nu$  appearing in the phase function  $\Phi(y, \nu)$ ; these parameters do not have to be related in any way, provided the estimates in hypotheses (I2) and (I4) hold uniformly in the appropriate parameters. We will simply unite both sets of parameters and call this union  $\nu$  again.*

*Proof.* It is clear that (4.2) holds for  $0 \leq \lambda \leq 1$  since  $|I(\lambda, \nu)|$  is bounded for such  $\lambda$ .

Now, consider the case where  $\lambda \geq 1$ . Set  $y = \rho\omega + z$ , where  $\omega \in \mathbb{S}^{N-1}$  (using the convention that  $\mathbb{S}^0 = \{-1, 1\}$ ),  $\rho > 0$  and  $z \in \mathbb{R}^N$  is some fixed point; then

$$I(\lambda, \nu) = \int_{\mathbb{S}^{N-1}} \int_0^\infty e^{i\lambda\Phi(\rho\omega+z, \nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) \rho^{N-1} d\rho d\omega.$$

By the compactness of  $\mathbb{S}^{N-1}$ , it suffices to prove (4.2) for the inner integral.

Choose a function  $\chi \in C_0^\infty([0, \infty))$ ,  $0 \leq \chi(s) \leq 1$  for all  $s$ , which is identically 1 on  $0 \leq s \leq \frac{1}{2}$  and is zero when  $s \geq 1$ ; then, writing  $F(\rho, \omega, \nu) = \Phi(\rho\omega + z, \nu)$ , we split the inner integral into the sum of the two integrals

$$\begin{aligned} I_1(\lambda, \nu, \omega, z) &= \int_0^\infty e^{i\lambda F(\rho, \omega, \nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) \chi(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1} d\rho, \\ I_2(\lambda, \nu, \omega, z) &= \int_0^\infty e^{i\lambda F(\rho, \omega, \nu)} A(\rho\omega + z, \nu) g(\rho\omega + z) (1 - \chi)(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1} d\rho. \end{aligned}$$

Let us first look at  $I_1 = I_1(\lambda, \nu, \omega, z)$ ; since  $\chi(\lambda^{\frac{1}{\gamma}} \rho)$  is zero for  $\lambda^{\frac{1}{\gamma}} \rho \geq 1$ , we have, by the change of variables  $\tilde{\rho} = \lambda^{\frac{1}{\gamma}} \rho$ ,

$$\begin{aligned} |I_1| &\leq C \int_0^\infty \chi(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1} d\rho = C \int_0^\infty (\tilde{\rho})^{N-1} \lambda^{-\frac{N-1}{\gamma}} \chi(\tilde{\rho}) \lambda^{-\frac{1}{\gamma}} d\tilde{\rho} \\ &\leq C \lambda^{-\frac{N}{\gamma}} \int_0^1 (\tilde{\rho})^{N-1} d\tilde{\rho} = C \lambda^{-\frac{N}{\gamma}}, \end{aligned}$$

where we have used  $|e^{i\lambda F(\rho, \omega, \nu)}| \leq 1$  since  $\text{Im } F(\rho, \omega, \nu) \geq 0$  for all  $\rho, \omega, \nu$  by hypothesis (I2); this is the desired estimate for  $|I_1|$ .

In order to estimate  $I_2 = I_2(\lambda, \nu, \omega, z)$ , let us first define the operator  $L := (i\lambda \partial_\rho F(\rho, \omega, \nu))^{-1} \frac{\partial}{\partial \rho}$  and observe that

$$L(e^{i\lambda F(\rho, \omega, \nu)}) = e^{i\lambda F(\rho, \omega, \nu)}.$$

Denoting the adjoint of  $L$  by  $L^*$ , we have, for each  $l \in \mathbb{N} \cup \{0\}$ ,

$$I_2 = \int_0^\infty e^{i\lambda F(\rho, \omega, \nu)} (L^*)^l [A(\rho\omega + z, \nu) g(\rho\omega + z) (1 - \chi)(\lambda^{\frac{1}{\gamma}} \rho) \rho^{N-1}] d\rho.$$

Now,

$$(L^*)^l = \left(\frac{i}{\lambda}\right)^l \sum C_{s_1, \dots, s_p, p, r, l} \frac{\partial_\rho^{s_1} F \dots \partial_\rho^{s_p} F}{(\partial_\rho F)^{l+p}}(\rho, \omega, \nu) \frac{\partial^r}{\partial \rho^r},$$

where the sum is over all integers  $s_1, \dots, s_p, p, r \geq 0$  such that  $s_1 + \dots + s_p + r - p = l$ . By Hypothesis (I3),

$$\left| \frac{\partial_\rho^{s_1} F \dots \partial_\rho^{s_p} F}{(\partial_\rho F)^{l+p}}(\rho, \omega, \nu) \right| \leq C \rho^{p-s_1-\dots-s_p-l\gamma+l} = C \rho^{r-l\gamma}.$$

Also, we claim that, for  $r \leq [\frac{N}{\gamma}] + 1$ ,

$$\left| \frac{\partial^r}{\partial \rho^r} [A(\rho\omega + z, \nu)g(\rho\omega + z)(1 - \chi)(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}] \right| \leq C_N \rho^{N-1-r} \tilde{\chi}(\lambda, \rho), \quad (4.3)$$

where  $\tilde{\chi}(\lambda, \rho)$  is a smooth function in  $\rho$  which is zero for  $\lambda^{\frac{1}{\gamma}}\rho < \frac{1}{2}$ . Assuming this is true, we see that, for large enough  $l$ —it suffices to take  $l = [\frac{N}{\gamma}] + 1$ , i.e.  $N - l\gamma < 0$ —we have,

$$\begin{aligned} |I_2| &\leq C_N \lambda^{-l} \int_0^\infty \sum C_{s_1, \dots, s_p, p, r, l} \rho^{r-l\gamma} [\rho^{N-1-r}] \tilde{\chi}(\lambda, \rho) d\rho \\ &\leq C_N \lambda^{-l} \int_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty \rho^{N-1-l\gamma} d\rho = C_N \lambda^{-l} \left[ \frac{\rho^{N-l\gamma}}{N-l\gamma} \right]_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty = C_{N,\gamma} \lambda^{-\frac{N}{\gamma}}; \end{aligned}$$

together with the estimate for  $|I_1|$ , this yields the desired estimate (4.2). Here we need  $l > N/\gamma$ , which means an application of  $(L^*)^l$ , or estimates on  $\partial_\rho^\alpha F$  for  $|\alpha| \leq l$ . This gives a restriction on the number  $m$  of derivatives in (I3).

Finally, let us check (4.3). It holds because:

- (i)  $|\partial_\rho^r(\rho^{N-1})| \leq C_{r,N} \rho^{N-1-r}$  for all  $r \in \mathbb{N}$ .
- (ii) For each  $r \in \mathbb{N}$ ,  $\partial_\rho^r[(1 - \chi)(\lambda^{\frac{1}{\gamma}}\rho)] = -\lambda^{\frac{r}{\gamma}}(\partial_s^r \chi)(\lambda^{\frac{1}{\gamma}}\rho)$ ; now,  $(\partial_s \chi)(\lambda^{\frac{1}{\gamma}}\rho)$  is supported on the set  $\left\{ (\lambda, \rho) \in (0, \infty) \times (0, \infty) : \frac{1}{2} < \lambda^{\frac{1}{\gamma}}\rho < 1 \right\}$ , so, in particular, on its support  $\lambda^{\frac{1}{\gamma}} < \rho^{-1}$ ; therefore,

$$|\partial_\rho^r[(1 - \chi)(\lambda^{\frac{1}{\gamma}}\rho)]| \leq C \rho^{-r} (\partial_s^r \chi)(\lambda^{\frac{1}{\gamma}}\rho) \quad \text{for all } r \in \mathbb{N},$$

and  $(\partial_s^r \chi)(\lambda^{\frac{1}{\gamma}}\rho)$  is smooth in  $\rho$  and zero for  $\lambda^{\frac{1}{\gamma}}\rho \leq \frac{1}{2}$ .

- (iii) By hypothesis (I4),  $|\partial_\rho^r A(\rho\omega + z, \nu)| \leq C_r$  for each  $r \leq [\frac{N}{\gamma}] + 1$  (this can be seen for  $r = 1$  by noting that  $\partial_\rho A(\rho\omega + z, \nu) = \omega \cdot \nabla_y A(y, \nu)|_{y=\rho\omega+z}$ , and then for  $r \geq 2$  by calculating the higher derivatives). Also,  $g$  is smooth in  $U$ , so,  $|\partial_\rho^r [A(\rho\omega + z, \nu)g(\rho\omega + z)]| \leq C_r$  for  $r \leq [\frac{N}{\gamma}] + 1$ . Furthermore, by hypothesis (I1), there exists a constant  $\rho_0 > 0$  so that  $g(\rho\omega + z) = 0$  for  $\rho > \rho_0$ ; thus,  $\partial_\rho^r [A(\rho\omega + z, \nu)g(\rho\omega + z)]$  is zero for  $\rho > \rho_0$ ; hence, for  $r \leq [\frac{N}{\gamma}] + 1$ ,

$$|\partial_\rho^r [A(\rho\omega + z, \nu)g(\rho\omega + z)]| \leq C_r \rho_0^r \rho^{-r}.$$

This completes the proof of the claim, and thus the theorem.  $\square$

## 4.2 Functions of convex type

Hypothesis (I3) of Theorem 4.1 is sufficient for the result of the theorem to hold; however, it is often difficult to check. For this reason, we now introduce the concept of a function of convex type—a condition that is far simpler to verify—and show that for such functions, (I3) automatically holds.

**Definition 4.3.** Let  $F = F(\rho, v) : [0, \infty) \times \Upsilon \rightarrow \mathbb{C}$  be a function that is smooth in  $\rho$  for each fixed  $v \in \Upsilon$ , where  $\Upsilon$  is some parameter space. Write its  $M^{\text{th}}$  order Taylor expansion in  $\rho$  about 0 in the form

$$F(\rho, v) = \sum_{j=0}^M a_j(v) \rho^j + R_M(\rho, v), \quad (4.4)$$

where  $R_M(\rho, v) = \int_0^\rho \partial_s^{M+1} F(s, v) \frac{(\rho-s)^M}{M!} ds$  is the  $M^{\text{th}}$  remainder term.

We say that  $F$  is a function of convex type  $\gamma$  if, for some  $\gamma \in \mathbb{N}$ ,  $\gamma \geq 2$ , and for some  $\delta > 0$ , we have

(CT1)  $a_0(v) = a_1(v) = 0$  for all  $v \in \Upsilon$  (i.e. the Taylor expansion of  $F$  starts from order  $\geq 2$ );

(CT2) there exists a constant  $C > 0$  such that  $\sum_{j=2}^\gamma |a_j(v)| \geq C$  for all  $v \in \Upsilon$ ;

(CT3) for each  $v \in \Upsilon$ ,  $|\partial_\rho F(\rho, v)|$  is increasing in  $\rho$  for  $0 < \rho < \delta$ ;

(CT4) for each  $k \in \mathbb{N}$ ,  $\partial_\rho^k F(\rho, v)$  is bounded uniformly in  $0 < \rho < \delta$ ,  $v \in \Upsilon$ .

**Remark 4.4.** Note that, if  $F$  is real-valued, then (CT3) implies that we have either  $\partial_\rho^2 F(\rho, v) \geq 0$  for all  $0 < \rho < \delta$ , or  $\partial_\rho^2 F(\rho, v) \leq 0$  for all  $0 < \rho < \delta$ —this is because  $\partial_\rho F(0, v) = 0$ . This is the connection with convexity, hence the name of such functions.

Such functions have the following useful property:

**Lemma 4.5.** Let  $F(\rho, v)$  be a function of convex type  $\gamma$ . Then, for each sufficiently small  $0 < \delta \leq 1$  there exist constants  $C, C_m > 0$  such that

$$|\partial_\rho F(\rho, v)| \geq C \rho^{\gamma-1} \quad (4.5)$$

$$\text{and } |\partial_\rho^m F(\rho, v)| \leq C_m \rho^{1-m} |\partial_\rho F(\rho, v)| \quad (4.6)$$

for all  $0 < \rho < \delta$ ,  $v \in \Upsilon$  and  $m \in \mathbb{N}$ .

**Remark 4.6.** A version of this lemma appeared in [Sug94] for analytic functions without dependence on  $v$  and is based on Lemmas 3, 4 and 5 of Randol [Ran69] (which also appeared in Beals [Bea82], Lemmas 3.2, 3.3). Lemma 4.5 extends it to functions that are only smooth and which depend on an additional parameter, which will be necessary of our analysis. A limited regularity version of this lemma appeared in [Ruzh07]. The proof of lemma given here is based on estimating the remainder rather than on using the Cauchy's integral formula for analytic functions.

*Proof.* First, let us note that, for  $0 < \rho \leq 1$  we have, by (CT2),

$$\pi(\rho, v) := \sum_{j=2}^\gamma j |a_j(v)| \rho^{j-1} \geq C \rho^{\gamma-1}. \quad (4.7)$$

Thus, in order to prove (4.5), it suffices to show

$$|\partial_\rho F(\rho, v)| \geq C\pi(\rho, v) \quad \text{for all } 0 < \rho < \delta, v \in \Upsilon; \quad (4.8)$$

For  $1 \leq m \leq \gamma$ , we have, using (4.4),

$$\partial_\rho^m F(\rho, v) = \sum_{k=0}^{\gamma-m} \frac{(k+m)!}{k!} a_{k+m}(v) \rho^k + R_{m, \gamma-m}(\rho, v), \quad (4.9)$$

where  $R_{m, \gamma-m}(\rho, v) = \int_0^\rho \partial_\rho^{\gamma+1} F(s, v) \frac{(\rho-s)^{\gamma-m}}{(\gamma-m)!} ds$  is the remainder term of the  $(\gamma-m)^{\text{th}}$  Taylor expansion of  $\partial_\rho^m F(\rho, v)$ . By (CT4) and (4.7), we see

$$|R_{m, \gamma-m}(\rho, v)| \leq C_{\gamma, m} \rho^{\gamma+1-m} \leq C_{\gamma, m} \pi(\rho, v) \rho^{2-m} \quad \text{for } 0 < \rho < \delta. \quad (4.10)$$

Hence, for  $0 < \rho < \delta$ ,

$$\begin{aligned} |\partial_\rho F(\rho, v)| &= \left| \sum_{k=0}^{\gamma-1} (k+1) a_{k+1}(v) \rho^k + R_{1, \gamma-1}(\rho, v) \right| \\ &\geq \left| \sum_{j=2}^{\gamma} j a_j(v) \rho^{j-1} \right| - |R_{1, \gamma-1}(\rho, v)| \geq \left| \sum_{j=2}^{\gamma} j a_j(v) \rho^{j-1} \right| - C_\gamma \pi(\rho, v) \rho. \end{aligned}$$

Now, by (CT3),  $|\partial_\rho F(\rho, v)|$  is increasing in  $\rho$  for each  $v \in \Upsilon$  and, by (CT1),  $\partial_\rho F(0, v) = 0$ ; therefore,

$$\begin{aligned} |\partial_\rho F(\rho, v)| &= \max_{0 \leq \sigma \leq \rho} |\partial_\rho F(\sigma, v)| \\ &\geq \max_{0 \leq \sigma \leq \rho} \left| \sum_{j=2}^{\gamma} j a_j(v) \sigma^{j-1} \right| - \max_{0 \leq \sigma \leq \rho} C_\gamma \pi(\sigma, v) \sigma \\ &= \max_{0 \leq \sigma \leq 1} \left| \sum_{j=2}^{\gamma} j a_j(v) \rho^{j-1} \bar{\sigma}^{j-1} \right| - C_\gamma \pi(\rho, v) \rho, \end{aligned}$$

since  $\pi(\sigma, v) \sigma = \sum_{j=2}^{\gamma} j |a_j(v)| \sigma^j$  clearly achieves its maximum on  $0 \leq \sigma \leq \rho$  at  $\sigma = \rho$ . Noting that

$$\max_{0 \leq \bar{\sigma} \leq 1} \left| \sum_{j=1}^L z_j \bar{\sigma}^{j-1} \right| \quad \text{and} \quad \sum_{j=1}^L |z_j|$$

are norms on  $\mathbb{C}^L$  and, hence, are equivalent, we immediately get

$$\begin{aligned} |\partial_\rho F(\rho, v)| &\geq C \sum_{j=2}^{\gamma} j |a_j(v)| \rho^{j-1} - C_\gamma \pi(\rho, v) \rho \\ &\geq (C - C_\gamma \delta) \pi(\rho, v) = C_{\gamma, \delta} \pi(\rho, v), \end{aligned}$$

which completes the proof of (4.8).

To prove (4.6), we consider the cases  $1 \leq m \leq \gamma$  and  $m > \gamma$  separately. For  $m > \gamma$ , we have, by (CT4),

$$|\partial_\rho^m F(\rho, v)| \leq C_m \leq C_{m,\delta} \rho^{\gamma+1-m} \quad \text{for } 0 < \rho < \delta,$$

since  $\gamma + 1 - m \leq 0$ , and, thus,  $\rho^{\gamma+1-m} \geq \delta^{\gamma+1-m} > 0$ ; so, by (4.5), we have

$$|\partial_\rho^m F(\rho, v)| \leq C_{m,\delta} \rho^{2-m} |\partial_\rho F(\rho, v)| \quad \text{for } 0 < \rho < \delta, m > \gamma. \quad (4.11)$$

For  $1 \leq m \leq \gamma$ , we have the representation (4.9). It is clear that

$$\left| \sum_{k=0}^{\gamma-m} \frac{(k+m)!}{k!} a_{k+m}(v) \rho^k \right| \leq C_m \pi(\rho, v) \rho^{1-m},$$

which, together with (4.10) and (4.8), yields

$$|\partial_\rho^m F(\rho, v)| \leq C_{m,\delta} \rho^{1-m} |\partial_\rho F(\rho, v)| \quad \text{for } 0 < \rho < \delta, 1 \leq m \leq \gamma.$$

This, together with (4.11), completes the proof of (4.6) and, thus, the lemma.  $\square$

This lemma means we have the following alternative version of Theorem 4.1.

**Corollary 4.7.** *Hypothesis (I3) of Theorem 4.1 may be replaced by:*

(I3') *for some fixed  $z \in \mathbb{R}^N$ , the function  $F(\rho, \omega, \nu) := \Phi(\rho\omega + z, \nu)$  is a function of convex type  $\gamma$ , for some  $\gamma \in \mathbb{N}$ , in the sense of Definition 4.3 with  $(\omega, \nu) \in \mathbb{S}^{N-1} \times \mathcal{N} \equiv \Upsilon$ .*

### 4.3 Convexity condition for real-valued phase functions

Using the results of the previous two sections, we can now prove a series of results for which a so-called convexity condition holds; here we recall Definitions 2.5 and 2.6 from Section 2 and prove the basic result for real-valued functions. We recall that a smooth function  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to satisfy the *convexity condition* if  $\Sigma_\lambda$  is convex for each  $\lambda \in \mathbb{R}$  (and the empty set is considered to be convex). The *maximal order of contact* of a hypersurface  $\Sigma$  is defined as follows. Let  $\sigma \in \Sigma$ , and denote the tangent plane at  $\sigma$  by  $T_\sigma$ . Let  $P$  be a plane containing the normal to  $\Sigma$  at  $\sigma$  and denote the order of the contact between the line  $T_\sigma \cap P$  and the curve  $\Sigma \cap P$  by  $\gamma(\Sigma; \sigma, P)$ . Then we set

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_P \gamma(\Sigma; \sigma, P).$$

In the proof of Theorem 2.8 we will need a Besov space version of the estimate for the kernel. For this, let us introduce some useful notation for a family of cut-off functions  $g_R \in C_0^\infty(\mathbb{R}^n)$ ,  $R \in [0, \infty)$ : these functions will correspond to the cut-offs to annuli in the frequency space and we need to trace the dependence on the parameter  $R$ . Suppose  $g \in C_0^\infty(\mathbb{R}^n)$  is such that, for some constants  $c_0, c_1 \geq 0$ , it is supported in the set

$$\{\xi : c_0 < |\xi| < c_1\},$$



and let  $g_0 \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  be another (arbitrary) compactly supported function. Then, for  $R \geq 0$ , set

$$g_R(\xi) := \begin{cases} g(\xi/R) & \text{if } R \geq 1, \\ g_0(\xi) & \text{if } 0 \leq R < 1. \end{cases} \quad (4.12)$$

Now we can prove the main convexity theorem:

**Theorem 4.8.** *Suppose  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the convexity condition. Set  $\gamma := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau))$  and assume this is finite. Let  $a(\xi)$  be a symbol of order  $\frac{n-1}{\gamma} - n$  of type  $(1, 0)$  on  $\mathbb{R}^n$ ; furthermore, on  $\text{supp } a$ , we assume:*

(i) *for all multi-indices  $\alpha$  there exists a constant  $C_\alpha > 0$  such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$

(ii) *there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\tau(\xi)| \geq C|\xi|$ ;*

(iii) *there exists a constant  $C_0 > 0$  such that  $|\partial_\omega \tau(\lambda\omega)| \geq C_0$  for all  $\omega \in \mathbb{S}^{n-1}$ ,  $\lambda > 0$ ; in particular,  $|\nabla \tau(\xi)| \geq C_0$  for all  $\xi \in \mathbb{R}^n \setminus \{0\}$ ;*

(iv) *there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,*

$$\frac{1}{\lambda} \Sigma_\lambda(\tau) \equiv \frac{1}{\lambda} \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0).$$

Then, the following estimate holds for all  $R \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $t > 1$ :

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) d\xi \right| \leq C t^{-\frac{n-1}{\gamma}}, \quad (4.13)$$

where  $g_R(\xi)$  is as given in (4.12) and  $C > 0$  is independent of  $R$ .

**Remark 4.9.** *For an integral of this type with some specific compactly supported function,  $\chi \in C_0^\infty(\mathbb{R}^n)$  say, in place of  $g_R$ , we can just use the result for  $R = 0$ . In this way we obtain Corollary 2.9.*

*Proof.* We may assume throughout, without loss of generality, that either  $\tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$  or  $\tau(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$ . Indeed, hypothesis (ii) and the continuity of  $\tau$  ensure that either  $\tau(\xi)$  is positive for all  $|\xi| \geq M$  or negative for all  $|\xi| \geq M$ . In the case where  $\tau(\xi)$  is positive for all  $|\xi| \geq M$ , set

$$\tau_+(\xi) := \tau(\xi) + \min(0, \inf_{|\xi| < M} \tau(\xi)) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Now,  $\tau(\xi) - \tau_+(\xi)$  is a constant (in particular, it is independent of  $\xi$ ) and  $|e^{i[\tau(\xi) - \tau_+(\xi)]t}| = 1$ , so it suffices to show

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_+(\xi)t)} a(\xi) g_R(\xi) d\xi \right| \leq C t^{-\frac{n-1}{\gamma}}.$$

In the case where  $\tau(\xi)$  is negative for  $|\xi| \geq M$ , set  $\tilde{\tau}(\xi) := -\tau(\xi)$  and by similar reasoning to above, it is sufficient to show

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \tilde{\tau}_+(\xi)t)} a(\xi) g_R(\xi) d\xi \right| \leq C t^{-\frac{n-1}{\gamma}},$$

where  $-\tilde{\tau}_+(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$ .

We begin by dividing the integral into two parts: near to the wave-front set, i.e. points where  $\nabla_\xi[x \cdot \xi + \tau(\xi)t] = 0$ , and away from such points. To this end, we introduce a cut-off function  $\kappa \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq \kappa(y) \leq 1$ , which is identically 1 in the ball of radius  $r > 0$  (which will be fixed below) centred at the origin,  $B_r(0)$ , and identically 0 outside the ball of radius  $2r$ ,  $B_{2r}(0)$ . Then we estimate the following two integrals separately:

$$\begin{aligned} I_1(t, x) &:= \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) \kappa(t^{-1}x + \nabla\tau(\xi)) d\xi, \\ I_2(t, x) &:= \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) (1 - \kappa)(t^{-1}x + \nabla\tau(\xi)) d\xi. \end{aligned}$$

For  $I_2(t, x)$  we have the following result:

**Lemma 4.10.** *Suppose  $a(\xi)$  is a symbol of order  $j \in \mathbb{R}$ . Then, for each  $l \in \mathbb{N}$  with  $l > n + j$ , we have, for all  $t > 0$ ,*

$$|I_2(t, x)| \leq C_{r,l} t^{-l}, \quad (4.14)$$

where the constants  $C_{r,l} > 0$  are independent of  $R$ .

*Proof.* In the support of  $(1 - \kappa)(t^{-1}x + \nabla\tau(\xi))$ , we have  $|x + t\nabla\tau(\xi)| \geq rt > 0$ , so we can write

$$\frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} \cdot \nabla_\xi(e^{i(x \cdot \xi + \tau(\xi)t)}) = e^{i(x \cdot \xi + \tau(\xi)t)};$$

therefore, denoting the adjoint to  $P \equiv \frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} \cdot \nabla_\xi$  by  $P^*$ , we get

$$I_2(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} (P^*)^l [a(\xi) g_R(\xi) (1 - \kappa)(t^{-1}x + \nabla\tau(\xi))] d\xi$$

for each  $l \in \mathbb{N}$ . We claim that for each  $l$  there exists some constant  $C_{r,l} > 0$  independent of  $R$  so that, when  $t > 1$ , we have

$$(P^*)^l [a(\xi) g_R(\xi) (1 - \kappa)(t^{-1}x + \nabla\tau(\xi))] \leq C_{r,l} t^{-l} (1 + |\xi|)^{j-l}; \quad (4.15)$$

assuming this, we obtain,

$$|I_2(t, x)| \leq C_{r,l} t^{-l} \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|)^{l-j}} d\xi.$$

Noting that  $\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|)^{l-j}} d\xi$  converges for  $l - j > n$  yields the desired estimate (4.14).

It remains to prove (4.15). Let  $f \equiv f(\xi; x, t)$  be a function that is zero for  $|x + t\nabla\tau(\xi)| \leq rt$  and is continuously differentiable with respect to  $\xi$ ; then,

$$P^*f = \nabla_\xi \cdot \left[ \frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} f \right] = \frac{t\Delta\tau(\xi)}{i|x + t\nabla\tau(\xi)|^2} f + \frac{(x + t\nabla\tau(\xi))}{i|x + t\nabla\tau(\xi)|^2} \cdot \nabla_\xi f - \frac{2t(x + t\nabla\tau(\xi)) \cdot [\nabla^2\tau(\xi) \cdot (x + t\nabla\tau(\xi))]}{i|x + t\nabla\tau(\xi)|^4} f. \quad (4.16)$$

Hence, using  $|x + t\nabla\tau(\xi)| \geq rt$  (hypothesis on  $f$ ) and  $|\partial^\alpha\tau(\xi)| \leq C(1 + |\xi|)^{1-|\alpha|}$  (hypothesis (i)), we have

$$|P^*f| \leq C_r t^{-1} [(1 + |\xi|)^{-1} |f| + |\nabla_\xi f|]. \quad (4.17)$$

Now, for all multi-indices  $\alpha$  and for all  $\xi \in \mathbb{R}^n$ , we get

- $|\partial^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{j-|\alpha|}$  for all  $\xi \in \mathbb{R}^n$  as  $a \in S_{1,0}^j(\mathbb{R}^n)$ ;
- $|\partial_\xi^\alpha [(1 - \kappa)(t^{-1}x + \nabla\tau(\xi))]| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}$ , for all  $\xi \in \mathbb{R}^n$ —here we have used hypothesis (i) once more. Also, it is zero for each  $\alpha$  when  $|x + t\nabla\tau(\xi)| \leq rt$  by the definition of  $\kappa$ .

Furthermore,  $|\partial^\alpha g_R(\xi)| = |\partial^\alpha g_0(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}$  for  $0 \leq R < 1$ , since  $C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset S_{1,0}^0(\mathbb{R}^n)$ . For  $R \geq 1$ , we have:

$$\begin{aligned} \partial^\alpha g_R(\xi) &= \partial^\alpha [g(\xi/R)] = R^{-|\alpha|} (\partial^\alpha g)(\xi/R) \text{ and } g \in S_{1,0}^0(\mathbb{R}^n) \\ \implies |\partial^\alpha g_R(\xi)| &\leq C_\alpha R^{-|\alpha|} (1 + |\xi/R|)^{-|\alpha|} \leq C_\alpha (1 + |\xi|)^{-|\alpha|}. \end{aligned}$$

Therefore,

$$|\partial^\alpha g_R(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|} \text{ for all } \xi \in \mathbb{R}^n \text{ and multi-indices } \alpha, \quad (4.18)$$

where the  $C_\alpha > 0$  are independent of  $R$ .

Hence, by (4.17), we obtain

$$|P^*[a(\xi)g_R(\xi)(1 - \kappa)(t^{-1}x + \nabla\tau(\xi))]| \leq C_r t^{-1} (1 + |\xi|)^{j-1}.$$

To prove (4.15) for  $l \geq 2$  we do induction on  $l$ . Note that

$$|(P^*)^l f| \leq C_r t^{-1} [(1 + |\xi|)^{-1} |(P^*)^{l-1} f| + |\nabla_\xi \{(P^*)^{l-1} f\}|].$$

The first term satisfies the desired estimate by the inductive hypothesis. For the second term, repeated application of the properties of  $a(\xi)$ ,  $g(\xi)$  and  $(1 - \kappa)(t^{-1}x + \nabla\tau(\xi))$  noted above to inductively estimate derivatives of  $(P^*)^{l'} f$ ,  $1 \leq l' \leq l - 2$  yields the desired estimate. This completes the proof of the lemma.  $\square$

This lemma, with  $j = \frac{n-1}{\gamma} - n$ , means that it suffices to prove (4.13) for  $I_1(t, x)$ , where  $|t^{-1}x + \nabla\tau(\xi)| < 2r$ .

Let  $\{\Psi_\ell(\xi)\}_{\ell=1}^L$  be a partition of unity in  $\mathbb{R}^n$  where  $\Psi_\ell(\xi) \in C^\infty(\mathbb{R}^n)$  is supported in a narrow (the breadth will be fixed below) open cone  $K_\ell$ ,  $\ell = 1, \dots, L$ ; let us assume that  $K_1$  contains the point  $e_n = (0, \dots, 0, 1)$  (if necessary, relabel the cones to ensure this) and also that each  $K_\ell$ ,  $\ell = 1, \dots, L$ , can be mapped onto  $K_1$  by rotation. Then, it suffices to estimate

$$I'_1(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa(t^{-1}x + \nabla\tau(\xi)) d\xi, \quad (4.19)$$

since the properties of  $\tau(\xi)$ ,  $a(\xi)$ ,  $g_R(\xi)$  and  $\kappa(t^{-1}x + \nabla\tau(\xi))$  used throughout are invariant under rotation.

By hypothesis (iii), the level sets  $\Sigma_\lambda = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$  are all non-degenerate (or empty). Furthermore, the Implicit Function Theorem allows us to parameterise the intersection of the surface  $\Sigma'_\lambda \equiv \frac{1}{\lambda}\Sigma_\lambda$  and the cone  $K_1$ :

$$K_1 \cap \Sigma'_\lambda = \{(y, h_\lambda(y)) : y \in U\};$$

here  $U \subset \mathbb{R}^{n-1}$  is a bounded open set for which  $p(U) = \mathbb{S}^{n-1} \cap K_1$  where  $p(y) = (y, \sqrt{1 - |y|^2})$ , and  $h_\lambda : U \rightarrow \mathbb{R}$  is a smooth function for each  $\lambda > 0$ ; in particular, each  $h_\lambda$  is concave due to  $\tau(\xi)$  satisfying the convexity condition, i.e.  $\Sigma'_\lambda$  is convex for each  $\lambda \in \mathbb{R}$ . Then, in the case that  $\tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$ , the cone  $K_1$  is parameterised by

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0, y \in U\},$$

and when  $\tau(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$ ,

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0, y \in U\}.$$

Now, let  $\underline{n} : K_1 \cap \Sigma'_\lambda \rightarrow \mathbb{S}^{n-1}$  be the Gauss map,

$$\underline{n}(\zeta) = \frac{\nabla\tau(\zeta)}{|\nabla\tau(\zeta)|}.$$

By the definition of  $\kappa(t^{-1}x + \nabla\tau(\xi))$ , we have

$$|t^{-1}x - (-\nabla\tau(\xi_\lambda))| < 2r$$

for each  $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$  that is also in the support of the integrand of (4.19). Hence, provided  $r > 0$  is taken sufficiently small, the convexity of  $\Sigma'_\lambda$  ensures that the points  $t^{-1}x/|t^{-1}x|$  and  $-\underline{n}(\xi_\lambda)$  are close enough so that there exists  $z(\lambda) \in U$  (for each  $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$ ) satisfying

$$\underline{n}(z(\lambda), h_\lambda(z(\lambda))) = -t^{-1}x/|t^{-1}x| = -x/|x| \in \mathbb{S}^{n-1}.$$

Also,  $(-\nabla_y h_\lambda(y), 1)$  is normal to  $\Sigma'_\lambda$  at  $(y, h_\lambda(y))$ , so, writing  $x = (x', x_n)$ , we have

$$-\frac{x}{|x|} = \frac{(-\nabla_y h_\lambda(z(\lambda)), 1)}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} \implies -\frac{x_n}{|x|} = \frac{1}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|}$$

$$\text{and } -\frac{x'}{|x|} = \frac{-\nabla_y h_\lambda(z(\lambda))}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} = \frac{x_n \nabla_y h_\lambda(z(\lambda))}{|x|};$$

therefore,  $-x' = x_n \nabla_y h_\lambda(z(\lambda))$ . We claim that  $x_n$  is away from 0 provided the breadth of the cone  $K_1$  is chosen to be sufficiently narrow, so

$$\frac{x'}{x_n} = -\nabla_y h_\lambda(z(\lambda)). \quad (4.20)$$

To prove this claim, first recall that  $\Sigma'_\lambda \subset B_{R_1}(0)$  for all  $\lambda > 0$  (hypothesis (iv)) and note that  $\partial_{\xi_n} \tau(\xi)$  is absolutely continuous on  $\overline{B_{R_1}(0)}$  (it is continuous in  $\mathbb{R}^n$ ): taking  $C_0 > 0$  as in hypothesis (iii), we get that

$$\begin{aligned} \text{there exists } \delta > 0 \text{ so that } |\eta^1 - \eta^2| < \delta, \text{ where } \eta^1, \eta^2 \in \overline{B_{R_1}(0)}, \\ \text{implies } |\partial_{\xi_n} \tau(\eta^1) - \partial_{\xi_n} \tau(\eta^2)| < C_0/4. \end{aligned} \quad (4.21)$$

Then, fix the breadth of  $K_1$  so that the maximal shortest distance from a point  $\xi \in K_1 \cap (\bigcup_{\lambda>0} \Sigma'_\lambda)$  to the ray  $\{\mu e_n : \mu > 0\}$  is less than this  $\delta$ , i.e.

$$\sup \left\{ \inf_{\mu>0} |\xi - \mu e_n| : \xi \in K_1 \cap \left( \bigcup_{\lambda>0} \Sigma'_\lambda \right) \right\} < \delta.$$

Now, observe that for any  $\xi^0 \in \mathbb{R}^n$ ,  $\mu > 0$ , we have

$$\left| \frac{x_n}{t} \right| \geq |\partial_{\xi_n} \tau(\mu e_n)| - |\partial_{\xi_n} \tau(\xi^0) - \partial_{\xi_n} \tau(\mu e_n)| - \left| \frac{x_n}{t} + \partial_{\xi_n} \tau(\xi^0) \right|.$$

Choose  $\xi^0 \in K_1 \cap \Sigma'_\lambda \cap \text{supp}[\kappa(t^{-1}x + \nabla \tau(\xi))]$  and  $\mu > 0$  so that  $|\xi^0 - \mu e_n| < \delta$  and, hence,

$$|\partial_{\xi_n} \tau(\xi^0) - \partial_{\xi_n} \tau(\mu e_n)| < C_0/4;$$

also, by hypothesis (iii),  $|\partial_{\xi_n} \tau(\mu e_n)| \geq C_0$ , so

$$|t^{-1}x_n| \geq 3C_0/4 - 2r.$$

Taking  $r$  sufficiently small, less than  $C_0/8$  say, (ensuring  $r > 0$  satisfies the earlier condition also) we get

$$|x_n| \geq ct > 0 \quad (4.22)$$

proving the claim.

Before estimating (4.19), we introduce some useful notation: by the definition of  $g_R(\xi)$ , (4.12), when  $R \geq 1$

$$\xi \in \text{supp } g_R \implies Rc_0 < |\xi| < Rc_1;$$

also, if  $0 \leq R < 1$ , then there exist constants  $\tilde{c}_0, \tilde{c}_1 > 0$  so that  $\tilde{c}_0 < |\xi| < \tilde{c}_1$  for  $\xi \in \text{supp } g_R$ . Thus, by hypotheses (i) and (ii), there exist constants  $c'_0, c'_1 > 0$  such that

$$\begin{cases} Rc'_0 < |\tau(\xi)| < Rc'_1 & \text{if } R \geq 1 \text{ and } \xi \in \text{supp } g_R, \\ c'_0 < |\tau(\xi)| < c'_1 & \text{if } 0 \leq R < 1 \text{ and } \xi \in \text{supp } g_R. \end{cases}$$

Let  $G \in C_0^\infty(\mathbb{R})$  be identically one on the set  $\{s \in \mathbb{R} : c'_0 < s < c'_1\}$  and identically zero in a neighbourhood of the origin; writing  $\mathcal{R} = \max(R, 1)$ , this then satisfies

$$g_R(\xi) = g_R(\xi)G(\tau(\xi)/\mathcal{R}).$$

Also, for simplicity, write

$$\tilde{a}(\xi) \equiv \tilde{a}_R(\xi) := a(\xi)g_R(\xi)\Psi_1(\xi); \quad (4.23)$$

this is a type (1,0) symbol of order  $\frac{n-1}{\gamma} - n$  supported in the cone  $K_1$ , and the constants in the symbolic estimates are all independent of  $R$  as each  $g_R(\xi)$ ,  $R \geq 0$ , is a symbol of order 0 with constants independent of  $R$  (see (4.18)).

We now turn to estimating (4.19). Using the change of variables  $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$  and equality (4.20), it becomes

$$\begin{aligned} I'_1(t, x) &= \int_0^\infty \int_U e^{i[\lambda x' \cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y)) \\ &\quad g_R(\lambda y, \lambda h_\lambda(y)) \Psi_1(\lambda y, \lambda h_\lambda(y)) \kappa(t^{-1}x + \nabla \tau(\lambda y, \lambda h_\lambda(y))) \frac{d\xi}{d(\lambda, y)} dy d\lambda \\ &= \int_0^\infty \int_U e^{i\lambda x_n [-\nabla_y h_\lambda(z(\lambda)) \cdot y + h_\lambda(y) + t x_n^{-1}]} \tilde{a}(\lambda y, \lambda h_\lambda(y)) \\ &\quad G(\lambda/\mathcal{R}) \kappa(t^{-1}x + \nabla \tau(\lambda y, \lambda h_\lambda(y))) \frac{d\xi}{d(\lambda, y)} dy d\lambda, \end{aligned} \quad (4.24)$$

where we have used  $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$  (definition of  $\Sigma_\lambda$ ) in the last line. Here, note that

$$\frac{d\xi}{d(\lambda, y)} = \begin{vmatrix} \lambda I & y \\ \lambda \nabla_y h_\lambda(y) & \partial_\lambda [\lambda h_\lambda(y)] \end{vmatrix} = \lambda^{n-1} (\partial_\lambda [\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)),$$

where  $I$  is the identity matrix. Differentiating  $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$  with respect to  $\lambda$  in the first case and with respect to  $y$  in the second, gives

$$\begin{aligned} y \cdot \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \partial_\lambda [\lambda h_\lambda(y)] \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) &= 1, \\ \lambda \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \lambda \nabla_y h_\lambda(y) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) &= 0. \end{aligned}$$

Substituting the second of these equalities into the first yields

$$(\partial_\lambda [\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 1.$$

We claim that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0. \quad (4.25)$$

To see this, first note that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq |\partial_{\xi_n} \tau(\lambda \mu e_n)| - |\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))|$$

where  $\mu > 0$  is chosen as above so that  $|\mu e_n - (y, h_\lambda(y))| \leq \delta$ ; now,  $|\partial_{\xi_n} \tau(\lambda \mu e_n)| \geq C_0$  by hypothesis (iii). Also, by the Mean Value Theorem, there exists  $\bar{\xi}$  lying on the segment between  $(\lambda y, \lambda h_\lambda(y))$  and  $\lambda \mu e_n$  such that

$$|\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \leq C |\nabla_\xi \partial_{\xi_n} \tau(\bar{\xi})| \lambda \delta \leq C |\bar{\xi}|^{-1} \lambda \delta \leq C \delta;$$

choosing  $\delta > 0$  small enough (also ensuring it satisfies condition (4.21) above) completes the proof of the claim. Hence,

$$\left| \frac{d\xi}{d(\lambda, y)} \right| = \left| \frac{\lambda^{n-1}}{\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))} \right| \leq C \lambda^{n-1}. \quad (4.26)$$

Also, note that this Jacobian is bounded below away from zero because  $|\partial_{\xi_n} \tau(\xi)| \leq C$  for all  $\xi \in \mathbb{R}^n$  (hypothesis (i)), which means that the transformation above is valid in  $K_1$ .

Next, using the change of variables  $\tilde{\lambda} = \lambda x_n = \lambda \tilde{x}_n t$  in (4.24), writing  $h(\lambda, y) \equiv h_\lambda(y)$  and setting  $\tilde{x} := t^{-1}x$  (so  $\tilde{x}_n = t^{-1}x_n$ ), we obtain

$$\begin{aligned} & \int_0^\infty \int_U e^{i\tilde{\lambda}(-\nabla_y h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z(\frac{\tilde{\lambda}}{\tilde{x}_n t}))) \cdot y + h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y) + \tilde{x}_n^{-1}} \tilde{a}\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right) \\ & G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \kappa\left(\tilde{x} + \nabla \tau\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right)\right) \frac{d\xi}{d(\lambda, y)} t^{-1} \tilde{x}_n^{-1} dy d\tilde{\lambda}. \end{aligned}$$

Therefore, using  $\left| \frac{d\xi}{d(\lambda, y)} \right| \leq C \tilde{\lambda}^{n-1} |\tilde{x}_n|^{-(n-1)} t^{-(n-1)}$  (by (4.26)) and recalling that  $|\kappa(\eta)| \leq 1$ , we have,

$$|I'_1(t, x)| \leq C t^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \left| I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; z\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}\right)\right) G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \tilde{\lambda}^{\frac{n-1}{\gamma}-1} \right| d\tilde{\lambda}, \quad (4.27)$$

where,

$$\begin{aligned} I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; z\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}\right)\right) &= \int_U e^{i\tilde{\lambda}[h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y) - h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z) - (y-z) \cdot \nabla_y h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z)]} \\ & \tilde{a}\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right) \left(\frac{\tilde{\lambda}}{t|\tilde{x}_n|}\right)^{n-\frac{n-1}{\gamma}} dy. \end{aligned}$$

With Theorem 4.1 in mind, let us rewrite this in the form of (4.1):

$$I(\lambda, \mu; z) = \int_{\mathbb{R}^{n-1}} e^{i\lambda \Phi(y, \mu; z)} a_0(\mu y, \mu h_\mu(y)) b(y) dy,$$

with arbitrary  $\lambda > 0$ ,  $\mu > 0$  and  $z \in \mathbb{R}^{n-1}$ , where

- $\Phi(y, \mu; z) = h_\mu(y) - h_\mu(z) - (y - z) \cdot \nabla_y h_\mu(z);$



- $a_0(\xi) := \tilde{a}(\xi)|\xi|^{n-\frac{n-1}{\gamma}};$
- $b \in C_0^\infty(\mathbb{R}^{n-1})$  with support contained in  $U$ .

We shall show that the following conditions (numbered as in Theorem 4.1 and Corollary 4.7) are satisfied by  $I(\lambda, \mu; z)$ :

- (I1) there exists a bounded set  $U \subset \mathbb{R}^{n-1}$  such that  $b \in C_0^\infty(U)$ ;
- (I2)  $\text{Im } \Phi(y, \mu; z) \geq 0$  for all  $y \in U, \mu > 0$ ;
- (I3')  $F(\rho, \omega, \mu; z) = \Phi(\rho\omega + z, \mu; z), \omega \in \mathbb{S}^{n-2}, \rho > 0$ , is a function of convex type  $\gamma$  (see Definition 4.3);
- (I4) there exist constants  $C_\alpha$  such that  $|\partial_y^\alpha [a_0(\mu y, \mu h_\mu(y))]| \leq C_\alpha$  for all  $y \in U, \mu > 0$  and  $|\alpha| \leq [\frac{n-1}{\gamma}] + 1$ .

Assuming for now that these hold, Theorem 4.1 (or, more precisely, Corollary 4.7) states that, for all  $\lambda > 0, \mu > 0$ ,

$$|I(\lambda, \mu; z)| \leq C(1 + \lambda)^{-\frac{n-1}{\gamma}} \leq C\lambda^{-\frac{n-1}{\gamma}}.$$

This, together with (4.27), gives

$$|I'_1(t, x)| \leq Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \tilde{\lambda}^{-\frac{n-1}{\gamma}} G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \tilde{\lambda}^{\frac{n-1}{\gamma}-1} d\tilde{\lambda};$$

then, setting  $\nu = \frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}$ , we have

$$\begin{aligned} |I'_1(t, x)| &\leq Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty (\mathcal{R}\tilde{x}_n t \nu)^{-1} G(\nu) \mathcal{R}\tilde{x}_n t d\nu \\ &= Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \nu^{-1} G(\nu) d\nu \leq Ct^{-\frac{n-1}{\gamma}} \quad \text{for all } t > 1. \end{aligned}$$

Here we have used that  $G$  is identically zero in a neighbourhood of the origin and that it is compactly supported and also (4.22) ( $|\tilde{x}_n| \geq C > 0$ ); also, note the constant here is independent of  $R$ . Since this inequality holds for  $I'_1(t, x)$ , it also holds for  $I_1(t, x)$ ; thus, together with Lemma 4.10, this proves the desired estimate (4.13), provided we show that the four properties (I1)–(I4) above hold.

Now, clearly (I1) holds automatically and (I2) is true since  $h_\mu(y)$  is real-valued, so  $\text{Im } \Phi(y, \mu; z) = 0$  for all  $y \in U, \mu > 0$ .

For (I3') and (I4), we need an auxiliary result about the boundedness of the derivatives of  $h_\lambda(y)$ :

**Lemma 4.11.** *All derivatives of  $h_\lambda(y)$  with respect to  $y$  are bounded uniformly in  $y$ . That is, for each multi-index  $\alpha$  there exists a constant  $C_\alpha > 0$  such that*

$$|\partial_y^\alpha h_\lambda(y)| \leq C_\alpha \quad \text{for all } y \in U, \lambda > 0.$$

*Proof.* By definition,  $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ . So,

$$(\nabla_{\xi'} \tau)(\lambda y, \lambda h_\lambda(y)) + (\partial_{\xi_n} \tau)(\lambda y, \lambda h_\lambda(y)) \nabla_y h_\lambda(y) = \lambda^{-1} \nabla_y [\tau(\lambda y, \lambda h_\lambda(y))] = 0,$$

or, equivalently,

$$\nabla_y h_\lambda(y) = - \frac{(\nabla_{\xi'} \tau)(\lambda y, \lambda h_\lambda(y))}{(\partial_{\xi_n} \tau)(\lambda y, \lambda h_\lambda(y))}. \quad (4.28)$$

Hypothesis (i) ( $|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1+|\xi|)^{1-|\alpha|}$  for all  $\xi \in \mathbb{R}^n$ ) and (4.25) ( $|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0$ ) then ensure that  $|\nabla_y h_\lambda(y)| \leq C$  for all  $y \in U$ ,  $\lambda > 0$ .

For higher derivatives, note that  $|(y, h_\lambda(y))| \leq R_1$  by hypothesis (iv); so, using hypothesis (i) once more, for all multi-indices  $\alpha$ , there exists a constant  $C_\alpha > 0$  such that

$$|(\partial_\xi^\alpha \tau)(\lambda y, \lambda h_\lambda(y))| \leq C_\alpha \lambda^{1-|\alpha|}.$$

Then, differentiating (4.28), this ensures, by an inductive argument, that the desired result for higher derivatives of  $h_\lambda(y)$  holds, proving the Lemma.  $\square$

Returning to the proof of (I4), note that,

$$|\partial_\xi^\alpha a_0(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|} \text{ for all } \xi \in \mathbb{R}^n,$$

since,  $\tilde{a}(\xi)$  is a symbol of order  $\frac{n-1}{\gamma} - n$  (see (4.23) for its definition). Together with Lemma 4.11, this ensures that  $\partial_y^\alpha [a_0(\mu y, \mu h_\mu(y))]$  is uniformly bounded for all  $y \in U$ ,  $\mu > 0$  and  $|\alpha| \leq \lceil \frac{n-1}{\gamma} \rceil + 1$  as required.

Finally, we show (I3'): observe that for  $|\rho| < \delta'$ , some suitably small  $\delta' > 0$ ,

$$\begin{aligned} F(\rho, \omega, \mu; z) &= h_\mu(\rho\omega + z) - h_\mu(z) - \rho\omega \cdot \nabla_y h_\mu(z) \\ &= \sum_{k=2}^{\gamma+1} \left[ \sum_{|\alpha|=k} \frac{1}{\alpha!} (\partial_y^\alpha h_\mu)(z) \omega^\alpha \right] \rho^k + R_{\gamma+1}(\bar{\rho}, \omega, \mu; z) \rho^{\gamma+2}. \end{aligned}$$

So,  $F(\rho, \omega, \mu; z)$  is a function of convex type  $\gamma$  if (using the numbering of Definition 4.3)

$$(CT2) \quad \sum_{k=2}^{\gamma+1} \left| \sum_{|\alpha|=k} \frac{1}{\alpha!} (\partial_y^\alpha h_\mu)(z) \omega^\alpha \right| \geq C > 0 \text{ for all } \omega \in \mathbb{S}^{n-2}, \mu > 0, z \in \mathbb{R}^{n-1}.$$

$$(CT3) \quad |\partial_\rho F(\rho, \omega, \mu; z)| \text{ is increasing in } \rho \text{ for } 0 < \rho < \delta, \text{ for each } \omega \in \mathbb{S}^{n-2}, \mu > 0;$$

$$(CT4) \quad \text{for each } k \in \mathbb{N}, \partial_\rho^k F(\rho, \omega, \mu; z) \text{ is bounded uniformly in } 0 < \rho < \delta', \omega \in \mathbb{S}^{n-2}, \mu > 0.$$

Condition (CT4), follows straight from Lemma 4.11. The concavity of  $h_\mu(y)$  means that

$$\partial_\rho^2 F(\rho, \omega, \mu; z) = \partial_\rho^2 [h_\mu(\rho\omega + z)] = \omega^t \text{Hess } h_\mu(\rho\omega + z) \omega \leq 0$$

for all  $0 < \rho < \delta'$  and for each  $\omega \in \mathbb{S}^{n-2}$ ,  $\mu > 0$ ,  $z \in \mathbb{R}^{n-1}$ ; coupled with the fact that  $\partial_\rho F(0, \omega, \mu; z) = 0$ , this ensures Condition (CT3) holds.

Lastly, recall that, by definition,  $\gamma \geq \gamma(\Sigma_\lambda)$  for all  $\lambda > 0$ , which is the maximal order of contact between  $\Sigma_\lambda$  and its tangent plane; furthermore,  $\gamma$  is assumed to be finite; thus, for some  $k \leq \gamma + 1 < \infty$ , we have

$$\partial_\rho^k [h_\mu(z + \rho\omega)]|_{\rho=0} \neq 0.$$

Now,  $\partial_\rho^k [h_\mu(z + \rho\omega)]|_{\rho=0} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial_y^\alpha h_\mu(z) \omega^\alpha$ , so for some  $k \leq \gamma + 1$ , we have

$$k! \left| \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial_y^\alpha h_\mu(z) \omega^\alpha \right| \geq C > 0$$

for all  $\omega \in \mathbb{S}^{n-2}$ . Thus, condition (CT2) holds.

This completes the proof of conditions (I1)–(I4), and, hence, Theorem 4.8.  $\square$

## 5 Oscillatory integrals without convexity

Theorem 4.8 requires the phase function to satisfy the convexity condition of Definition 2.5; however, we will also investigate solutions to hyperbolic equations for which the characteristic roots do not necessarily satisfy such a condition. In this section we state and prove a theorem for this case. First, we give the key results that replaces Theorem 4.1 in the proof, the well-known van der Corput Lemma. We recall the standard van der Corput Lemma as given in, for example, [Sog93, Lemma 1.1.2], or in [Ste93, Proposition 2, Ch VIII]:

**Lemma 5.1.** *Let  $\Phi \in C^\infty(\mathbb{R})$  be real-valued,  $a \in C_0^\infty(\mathbb{R})$  and  $m \geq 2$  be an integer such that  $\Phi^{(j)}(0) = 0$  for  $0 \leq j \leq m-1$  and  $\Phi^{(m)}(0) \neq 0$ ; then*

$$\left| \int_0^\infty e^{i\lambda\Phi(x)} a(x) dx \right| \leq C(1 + \lambda)^{-1/m} \quad \text{for all } \lambda \geq 0,$$

*provided the support of  $a$  is sufficiently small. The constant on the right-hand side is independent of  $\lambda$  and  $\Phi$ .*

If  $m = 1$ , then the same result holds provided  $\Phi'(x)$  is monotonic on the support of  $a$ .

### 5.1 Real-valued phase function

In the case when the convexity condition holds the estimate of Theorem 4.8 is given in terms of the constant  $\gamma$ ; as in the case of the homogeneous operators (see Introduction, Section 1.2) we introduce an analog to this in the case where the convexity condition does not hold. Let  $\Sigma$  be a hypersurface in  $\mathbb{R}^n$ ; we set

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_P \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma)$$

where  $\gamma(\Sigma; \sigma, P)$  is as in Definition 2.6.

An important result for calculating this value is the following:

**Lemma 5.2** ([Sug96]). Suppose  $\Sigma = \{(y, h(y)) : y \in U\}$ ,  $h \in C^\infty(U)$ ,  $U \subset \mathbb{R}^{n-1}$  is an open set, and let

$$F(\rho) = h(\eta + \rho\omega) - h(\eta) - \rho \nabla h(\eta) \cdot \omega$$

where  $\eta \in U$ ,  $\omega \in \mathbb{S}^{n-2}$ . Taking  $\sigma = (\eta, h(\eta)) \in \Sigma$ ,  $\omega \in \mathbb{S}^{n-2}$  and

$$P = \{\sigma + s(\omega, \nabla h(\eta) \cdot \omega) + t(-\nabla h(\eta), 1) \in \mathbb{R}^n : s, t \in \mathbb{R}\},$$

then

$$\gamma(\Sigma; \sigma, P) = \min \{k \in \mathbb{N} : F^{(k)}(0) \neq 0\} =: \gamma(h; \eta, \omega).$$

Therefore,

$$\begin{aligned} \gamma(\Sigma) &= \sup_{\eta} \sup_{\omega} \gamma(h; \eta, \omega), \\ \gamma_0(\Sigma) &= \sup_{\eta} \inf_{\omega} \gamma(h; \eta, \omega). \end{aligned}$$

Now we are in a position to state and prove the result for oscillatory integrals with a real-valued phase function that does not satisfy the earlier introduced convexity condition. This is a parameter dependent version of Corollary 2.13.

**Theorem 5.3.** Let  $a(\xi)$  be a symbol of order  $\frac{1}{\gamma_0} - n$  of type  $(1, 0)$  on  $\mathbb{R}^n$ . Let  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth on  $\text{supp } a$ , set  $\gamma_0 := \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\tau))$  and assume it is finite; furthermore, on  $\text{supp } a$ , we also assume the following conditions:

(i) for all multi-indices  $\alpha$  there exists a constant  $C_\alpha > 0$  such that

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$

(ii) there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\tau(\xi)| \geq C|\xi|$ ;

(iii) there exists a constant  $C_0 > 0$  such that  $|\partial_\omega \tau(\lambda\omega)| \geq C_0$  for all  $\omega \in \mathbb{S}^{n-1}$ ,  $\lambda > 0$ ;

(iv) there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,

$$\frac{1}{\lambda} \Sigma_\lambda(\tau) \subset B_{R_1}(0).$$

Then, the following estimate holds for all  $R \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $t > 1$ :

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) d\xi \right| \leq C t^{-\frac{1}{\gamma_0}},$$

where  $g_R(\xi)$  is as given in (4.12) and  $C > 0$  is independent of  $R$ .

*Proof.* We follow the proof of Theorem 4.8 as far as possible, and shall show how the absence of the convexity condition affects the estimate. Thus, as in the proof of Theorem 4.8, we may first assume, without loss of generality, that either  $\tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$  or  $\tau(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$ . We will always work on the support of  $a$ , so by writing  $\xi \in \mathbb{R}^n$  we will mean  $\xi \in \text{supp } a$ .

Divide the integral into two parts:

$$I_1(t, x) := \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) \kappa(t^{-1}x + \nabla \tau(\xi)) d\xi,$$

$$I_2(t, x) := \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) (1 - \kappa)(t^{-1}x + \nabla \tau(\xi)) d\xi,$$

where  $\kappa \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq \kappa(y) \leq 1$ , which is identically 1 in the ball of radius  $r > 0$  centred at the origin,  $B_r(0)$ , and identically 0 outside the ball of radius  $2r$ ,  $B_{2r}(0)$ . By Lemma 4.10 (which does not require the phase function to satisfy the convexity condition), we have

$$|I_2(t, x)| \leq C_r t^{-1/\gamma_0} \quad \text{for all } t > 1.$$

To estimate  $|I_1(t, x)|$  we introduce, as before, a partition of unity  $\{\Psi_\ell(\xi)\}_{\ell=1}^L$  and restrict attention to

$$I'_1(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa(t^{-1}x + \nabla \tau(\xi)) d\xi,$$

where  $\Psi_1(\xi)$  is supported in a sufficiently narrow cone,  $K_1$ , that contains  $e_n = (0, \dots, 0, 1)$ . Parameterise this cone in the same way as above: with  $U \subset \mathbb{R}^{n-1}$ ,

$$K_1 = \begin{cases} \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0, y \in U\} & \text{if } \tau(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^n \\ \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0, y \in U\} & \text{if } \tau(\xi) \leq 0 \text{ for all } \xi \in \mathbb{R}^n. \end{cases}$$

Here the Implicit Function Theorem ensures the existence of a smooth function  $h_\lambda : U \rightarrow \mathbb{R}$  for each  $\lambda > 0$ , but there is one major difference: the functions  $h_\lambda$  are not necessarily concave, in contrast to the earlier proof. Using the change of variables  $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$ —note that

$$0 < C \leq \left| \frac{d\xi}{d(\lambda, y)} \right| \leq C \lambda^{n-1}$$

by the same argument as in the proof of Theorem 4.8, providing the width of  $K_1$  is taken to be sufficiently small—gives

$$I'_1(t, x) = \int_0^\infty \int_U e^{i[\lambda x' \cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y)) g_R(\lambda y, \lambda h_\lambda(y)) \Psi_1(\lambda y, \lambda h_\lambda(y)) \kappa(t^{-1}x + \nabla \tau(\lambda y, \lambda h_\lambda(y))) \frac{d\xi}{d(\lambda, y)} dy d\lambda.$$

Once again, let  $G \in C_0^\infty(\mathbb{R})$  so that  $g_R(\xi) = g_R(\xi) G(\tau(\xi)/\mathcal{R})$  (where  $\mathcal{R} = \max(R, 1)$ ) and  $\tilde{a}(\xi) = a(\xi) g_R(\xi) \Psi_1(\xi)$ , which is a symbol of order  $\frac{1}{\gamma_0} - n$  supported in  $K_1$  and

with all the constants in the symbolic estimates independent of  $R$ . So, recalling that  $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$  and writing  $h(\lambda, y) \equiv h_\lambda(y)$ , we get

$$\begin{aligned} I'_1(t, x) &= \int_0^\infty \int_U e^{i\lambda[x' \cdot y + x_n h_\lambda(y) + t]} \tilde{a}(\lambda y, \lambda h_\lambda(y)) \\ &\quad G(\lambda/\mathcal{R}) \kappa(t^{-1}x + \nabla \tau(\lambda y, \lambda h_\lambda(y))) \frac{d\xi}{d(\lambda, y)} dy d\lambda \\ &= \int_0^\infty \int_U e^{i\tilde{\lambda}[\frac{\tilde{x}'}{\tilde{x}_n} \cdot y + h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y) + \tilde{x}_n^{-1}]} \tilde{a}\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right) \\ &\quad G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \kappa\left(\tilde{x} + \nabla \tau\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right)\right) \frac{d\xi}{d(\lambda, y)} \tilde{x}_n^{-1} t^{-1} dy d\tilde{\lambda}, \end{aligned}$$

where  $x = t\tilde{x}$  and  $\tilde{\lambda} = \lambda x_n = \lambda \tilde{x}_n t$ . Thus, using  $|\kappa(\eta)| \leq 1$ , we have

$$|I'_1(t, x)| \leq C |\tilde{x}_n|^{-1/\gamma_0} t^{-1/\gamma_0} \int_0^\infty \left| I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1} \tilde{x}\right) G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \tilde{\lambda}^{-1+(1/\gamma_0)} \right| d\tilde{\lambda} \quad (5.1)$$

where

$$I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1} \tilde{x}'\right) = \int_U e^{i\tilde{\lambda}[\tilde{x}_n^{-1} \tilde{x}' \cdot y + h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y)]} \tilde{a}\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right) \left(\frac{\tilde{\lambda}}{|\tilde{x}_n|t}\right)^{n-\frac{1}{\gamma_0}} dy.$$

At this point, we diverge from the proof of the earlier theorem since we cannot apply Theorem 4.1; instead, note that, for some  $b \in C_0^\infty(\mathbb{R}^{n-1})$  with support contained in  $U$ , we have

$$\begin{aligned} \left| I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1} \tilde{x}'\right) \right| &\leq \int_{\mathbb{R}^{n-2}} \left| \int_{\mathbb{R}} e^{i\tilde{\lambda}[\tilde{x}_n^{-1} \tilde{x}' \cdot y + h(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y)]} \right. \\ &\quad \left. \tilde{a}\left(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right) \left(\frac{\tilde{\lambda}}{|\tilde{x}_n|t}\right)^{n-\frac{1}{\gamma_0}} b(y) dy_1 \right| dy'. \end{aligned}$$

We wish to apply the van der Corput Lemma, Lemma 5.1, to the inner integral. Set  $\Phi(y, \mu; z) := z \cdot y + h_\mu(y)$ , which is real-valued, and consider the integral

$$\int_{\mathbb{R}} e^{i\lambda \Phi(y, \mu; z)} a_0(y, \mu) b(y) dy_1$$

where  $a_0(y, \mu) := \mu^{n-(1/\gamma_0)} \tilde{a}(\mu y, \mu h_\mu(y))$ . Recall that

$$\Sigma_\mu = \{(y, h_\mu(y)) : y \in U\},$$

so by Lemma 5.2,

$$\min \left\{ k \in \mathbb{N} : \partial_{y_1}^k \Phi(y, \mu; z) \big|_{y_1=0} \neq 0 \right\} = \gamma(h_\mu; 0, (1, 0, \dots, 0)) =: m.$$

Fixing the size of  $U$  so that  $|\partial_{y_1}^{(m)} \Phi(y, \mu; z)| \geq \varepsilon > 0$  for all  $y \in U$  ensures that the hypotheses of Lemma 5.1 are satisfied. Thus, since the support of  $b$  is compact in  $\mathbb{R}^{n-1}$ , is contained in  $U$ , and  $a_0$  is smooth, we obtain

$$\left| \int_{\mathbb{R}} e^{i\lambda \Phi(y, \mu; z)} a_0(y, \mu) b(y) dy_1 \right| \leq C \lambda^{-1/m}.$$

Carry out a suitable change of coordinates so that  $m = \inf_{\omega} \gamma(h_{\mu}; 0, \omega)$  (this is possible due to the rotational invariance of all properties used); then, since  $m \leq \gamma_0$  by definition, we have

$$\left| I\left(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1} \tilde{x}' \right) \right| \leq C \tilde{\lambda}^{-1/\gamma_0},$$

for all  $\tilde{\lambda}$  such that  $\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t} \in \text{supp } G$  (this is to ensure  $\tilde{\lambda}$  is away from the origin). Combining this with (5.1) then gives the required estimate:

$$\begin{aligned} |I'_1(t, x)| &\leq C |\tilde{x}_n|^{-1/\gamma_0} t^{-1/\gamma_0} \int_0^\infty \left| \tilde{\lambda}^{-1} G\left(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\right) \right| d\tilde{\lambda} \\ &= C |\tilde{x}_n|^{-1/\gamma_0} t^{-1/\gamma_0} \int_0^\infty (\nu \mathcal{R}\tilde{x}_n t)^{-1} G(\nu) \mathcal{R}\tilde{x}_n t d\nu \leq C t^{-\frac{1}{\gamma_0}}. \end{aligned} \quad \square$$

## 6 Decay of solutions to the Cauchy problem

Recall that we begin with the Cauchy problem with solution  $u = u(t, x)$

$$\begin{cases} D_t^m u + \sum_{j=1}^m P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, & t > 0, \\ D_t^l u(0, x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), & l = 0, \dots, m-1, x \in \mathbb{R}^n, \end{cases} \quad (6.1)$$

where  $P_j(\xi)$ , the polynomial obtained from the operator  $P_j(D_x)$  by replacing each derivative  $D_{x_k} = \frac{1}{i} \partial_{x_k}$  by  $\xi_k$ , is a constant coefficient homogeneous polynomial of order  $j$ , and the  $c_{\alpha,r}$  are constants. In this section we will prove different parts of Theorem 2.18.

### 6.1 Representation of the solution

Applying the partial Fourier transform with respect to  $x$  yields an ordinary differential equation for  $\hat{u} = \hat{u}(t, \xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(t, x) dx$ :

$$D_t^m \hat{u} + \sum_{j=1}^m P_j(\xi) D_t^{m-j} \hat{u} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha D_t^r \hat{u} = 0, \quad (6.2a)$$

$$D_t^l \hat{u}(0, \xi) = \hat{f}_l(\xi), \quad l = 0, \dots, m-1, \quad (6.2b)$$

where  $(t, \xi) \in [0, \infty) \times \mathbb{R}^n$  and  $P_j(\xi)$  are symbols of  $P_j(D_x)$ . Let  $E_j = E_j(t, \xi)$ ,  $j = 0, \dots, m-1$ , be the solutions to (6.2a) with initial data

$$D_t^l E_j(0, \xi) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases} \quad (6.2c)$$

Then the solution  $u$  of (6.1) can be written in the form

$$u(t, x) = \sum_{j=0}^{m-1} (\mathcal{F}^{-1} E_j \mathcal{F} f_j)(t, x), \quad (6.3)$$



where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  represent the partial Fourier transform with respect to  $x$  and its inverse, respectively.

Now, as (6.2a), (6.2b) is the Cauchy problem for a linear ordinary differential equation, we can write, denoting the characteristic roots of (6.1) by  $\tau_1(\xi), \dots, \tau_m(\xi)$ ,

$$E_j(t, \xi) = \sum_{k=1}^m A_j^k(t, \xi) e^{i\tau_k(\xi)t},$$

where  $A_j^k(t, \xi)$  are polynomials in  $t$  whose coefficients depend on  $\xi$ . Moreover, for each  $k = 1, \dots, m$  and  $j = 0, \dots, m-1$ , the  $A_j^k(t, \xi)$  are independent of  $t$  at points of the (open) set  $\{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \forall l \neq k\}$ ; when this is the case, we write  $A_j^k(t, \xi) \equiv A_j^k(\xi)$ . In particular, there exists  $M > 0$  such that if  $|\xi| \geq M$ , the roots are pairwise distinct. For  $A_j^k(\xi)$ , we have the following properties:

**Lemma 6.1.** *Suppose  $\xi \in S_k := \{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \forall l \neq k\}$ ; then we have the following formula:*

$$A_j^k(\xi) = \frac{(-1)^j \sum_{1 \leq s_1 < \dots < s_{m-j-1} \leq m} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\prod_{l=1, l \neq k}^m (\tau_l(\xi) - \tau_k(\xi))}, \quad (6.4)$$

where  $\sum^k$  means sum over the range indicated excluding  $k$ . Furthermore, we have, for each  $j = 0, \dots, m-1$  and  $k = 1, \dots, m$ ,

- (i)  $A_j^k(\xi)$  is smooth in  $S_k$ ;
- (ii)  $A_j^k(\xi) = O(|\xi|^{-j})$  as  $|\xi| \rightarrow \infty$ .

*Proof.* The representation (6.4) follows from Cramer's rule (and is done explicitly in [Kli67]):  $A_j^k(\xi) = \frac{\det V_j^k}{\det V}$ , where  $V := (\tau_i^{l-1}(\xi))_{i,l=1}^m$  is the Vandermonde matrix and  $V_j^k$  is the matrix obtained by taking  $V$  and replacing the  $k^{\text{th}}$  column by  $(\underbrace{0 \dots 0}_j 1 \ 0 \dots 0)^T$ .

Smoothness of  $A_j^k(\xi)$  then follows by Proposition 3.4 and the asymptotic behaviour is a consequence of Part I of Proposition 3.5 since (6.4) holds for all  $|\xi| \geq M$ .  $\square$

## 6.2 Division of the integral

We choose  $M > 0$  so that all roots  $\tau_k(\xi)$ ,  $k = 1, \dots, n$ , are distinct for  $|\xi| \geq M$ . Let  $\chi = \chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq \chi(\xi) \leq 1$ , be a cut-off function that is identically 1 for  $|\xi| < M$  and identically zero for  $|\xi| > 2M$ . Then (6.3) can be rewritten as:

$$u(t, x) = \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j \chi \mathcal{F} f_j)(t, x) + \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j (1 - \chi) \mathcal{F} f_j)(t, x). \quad (6.5)$$

**Large  $|\xi|$ :** The second term of (6.5) is the most straightforward to study: by the choice of  $M$ , we have

$$E_j(t, \xi)(1 - \chi)(\xi) = \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(\xi)(1 - \chi)(\xi);$$

therefore, since each summand is smooth in  $\mathbb{R}^n$ , we can write

$$\begin{aligned} \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1 - \chi)\mathcal{F}f_j)(t, x) \\ = \frac{1}{(2\pi)^n} \sum_{j=0}^{m-1} \sum_{k=1}^m \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi)(1 - \chi)(\xi) \widehat{f}_j(\xi) d\xi. \end{aligned}$$

Each of these integrals may be studied separately. Note that, unlike in the cases of the wave equation, Brenner [Bre75], and the general  $m^{\text{th}}$  order homogeneous strictly hyperbolic equations, Sugimoto [Sug94], we may not assume that  $t = 1$ . The  $L^p - L^q$  estimates obtained under different conditions on the phase function for operators of this type are given in Section 6.3 below.

**Bounded  $|\xi|$ :** We turn our attention to the terms of the first sum in (6.5), the case of bounded frequencies,

$$\mathcal{F}^{-1}(E_j \chi \mathcal{F}f)(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi. \quad (6.6)$$

Unlike in the case above, here the characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$  are not necessarily distinct at all points in the support of the integrand (which is contained in the ball of radius  $2M$  about the origin); in particular, this means that the  $A_j^k(t, \xi)$  may genuinely depend on  $t$  and we have no simple formula valid for them in the whole region.

For this reason, we begin by systematically separating neighbourhoods of points where roots meet—referred to henceforth as multiplicities—from the rest of the region, and then considering the two cases separately. In Section 6.9 we find  $L^p - L^q$  estimates in the region away from multiplicities under various conditions; in Section 7 we show how these differ in the neighbourhoods of singularities. First, we need to understand in what type of sets the roots  $\tau_k(\xi)$  can intersect:

**Lemma 6.2.** *The complement of the set of multiplicities of a linear strictly hyperbolic constant coefficient partial differential operator  $L(D_t, D_x)$ ,*

$$S := \{\xi \in \mathbb{R}^n : \tau_j(\xi) \neq \tau_k(\xi) \text{ for all } j \neq k\},$$

*is dense in  $\mathbb{R}^n$ .*

*Proof.* First note

$$S = \{\xi \in \mathbb{R}^n : \Delta_L(\xi) \neq 0\} ,$$

where  $\Delta_L$  is the discriminant of  $L(\tau, \xi)$  (see the proof of Proposition 3.4 for definition and some properties). Now, by Sylvester's Formula (see [GKZ94]),  $\Delta_L$  is a polynomial in the coefficients of  $L(\tau, \xi)$ , which are themselves polynomials in  $\xi$ . Hence,  $\Delta_L$  is a polynomial in  $\xi$ ; as it is not identically zero (for large  $|\xi|$ , the characteristic roots are distinct, and hence it is non-zero at such points), it cannot be zero on an open set, and hence its complement is dense in  $\mathbb{R}^n$ .  $\square$

**Corollary 6.3.** *Let  $L(D_t, D_x)$  be a linear strictly hyperbolic constant coefficient partial differential operator with characteristic roots  $\tau_1(\xi), \dots, \tau_m(\xi)$ . Suppose, for  $k \neq l$ , that  $\mathcal{M}_{kl} \subset \mathbb{R}^n$  is the set of all  $\xi$  such that  $\tau_k(\xi) = \tau_l(\xi)$ . For  $\varepsilon > 0$ , define*

$$\mathcal{M}_{kl}^\varepsilon := \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}_{kl}) < \varepsilon\} ;$$

*denote the largest  $\nu \in \mathbb{N}$  such that  $\text{meas}(\mathcal{M}_{kl}^\varepsilon) \leq C\varepsilon^\nu$  for all sufficiently small  $\varepsilon > 0$  by  $\text{codim } \mathcal{M}_{kl}$ . Then  $\text{codim } \mathcal{M}_{kl} \geq 1$ .*

*Proof.* Follows straight from Lemma 6.2: the fact that  $\mathcal{M}_{kl}$  has non-empty interior (it is an algebraic set) ensures that its  $\varepsilon$ -neighbourhood is bounded by  $C\varepsilon$  in at least one dimension for all small  $\varepsilon > 0$ .  $\square$

We can note that if  $L(D_t, D_x)$  is not differential, but pseudo-differential in  $D_x$ , the rest of the analysis goes through in a similar way, but we may need to assume that  $\text{codim } \mathcal{M}_{kl} \geq 1$ .

With this in mind, we shall subdivide the integral (6.6): suppose  $L$  roots meet in a set  $\mathcal{M}$  with  $\text{codim } \mathcal{M} = \ell$ ; without loss of generality, by relabelling, assume the coinciding roots are  $\tau_1(\xi), \dots, \tau_L(\xi)$ . By continuity, there exists an  $\varepsilon > 0$  such that they do not intersect other roots  $\tau_{L+1}, \dots, \tau_m$  in  $\mathcal{M}^\varepsilon$ . Furthermore, we may assume that  $\partial \mathcal{M}^\varepsilon \in C^1$ : for each  $\varepsilon > 0$  there exists a set  $S_\varepsilon$  with  $C^1$  boundary such that  $\mathcal{M}^\varepsilon \subset S_\varepsilon$  and  $\text{meas}(S_\varepsilon \setminus \mathcal{M}^\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then:

1. Let  $\chi_{\mathcal{M}, \varepsilon} \in C^\infty(\mathbb{R}^n)$  be a smooth function identically 1 on  $\mathcal{M}^\varepsilon$  and identically zero outside  $\mathcal{M}^{2\varepsilon}$ ; now consider the subdivision of (6.6):

$$\begin{aligned} \int_{B_{2M}(0)} e^{ix \cdot \xi} E_j(t, \xi) \widehat{f}(\xi) d\xi &= \int_{B_{2M}(0)} e^{ix \cdot \xi} E_j(t, \xi) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi \\ &\quad + \int_{B_{2M}(0)} e^{ix \cdot \xi} E_j(t, \xi) (1 - \chi_{\mathcal{M}, \varepsilon})(\xi) \widehat{f}(\xi) d\xi ; \end{aligned}$$

for the second integral, simply repeat the above procedure around any root multiplicities in  $B_{2M}(0) \setminus \mathcal{M}^\varepsilon$ .

2. For the first integral, the case where the integrand is supported on  $\mathcal{M}^\varepsilon$ , split off the coinciding roots from the others:

$$\begin{aligned} & \int_{B_{2M}(0)} e^{ix \cdot \xi} E_j(t, \xi) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi \\ &= \int_{B_{2M}(0)} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi \\ & \quad + \int_{B_{2M}(0)} e^{ix \cdot \xi} \left( \sum_{k=L+1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi. \quad (6.7) \end{aligned}$$

3. For the first integral, we use techniques discussed in Section 7 below to estimate it.
4. For the second there are two possibilities: firstly, two or more of the characteristic roots  $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$  coincide in  $B_{2M}(0)$ —in this case, repeat the procedure above for this integral. Alternatively, these roots are all distinct in  $B_{2M}(0) \setminus \mathcal{M}^\varepsilon$ —in this case, it suffices to study each integral separately as the  $A_j^k(t, \xi)$  are independent of  $t$ , and thus the expression (6.4) is valid and we can write

$$\begin{aligned} & \int_{B_{2M}(0)} e^{ix \cdot \xi} \left( \sum_{k=L+1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi \\ &= \sum_{k=L+1}^m \int_{B_{2M}(0)} e^{i[x \cdot \xi + \tau_k(\xi)t]} A_j^k(\xi) \chi_{\mathcal{M}, \varepsilon}(\xi) \widehat{f}(\xi) d\xi; \end{aligned}$$

estimates for integrals of the type on the right-hand side are found in Section 6.9—note that in this case we may use that the region is bounded to ensure that all continuous functions are also bounded.

Continue this procedure until all multiplicities are accounted for in this way.

Finally, let us recall the following result that can be found in [BL76, Theorem 6.4.5]:

**Theorem 6.4.** *Suppose  $T$  is a linear map such that it maps*

$$T : W_{p_0}^{s_0} \rightarrow L^{q_0}, \quad T : W_{p_1}^{s_1} \rightarrow L^{q_1},$$

where  $s_0 \neq s_1$ ,  $1 \leq p_0, p_1 < \infty$ ; then  $T$  also maps:

$$T : W_{p_\theta}^{s_\theta} \rightarrow L^{q_\theta},$$

where  $0 \leq \theta \leq 1$  and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s_\theta = (1-\theta)s_0 + \theta s_1.$$

That is,  $\|Tf\|_{L^{q_\theta}} \leq C \|f\|_{W_{p_\theta}^{s_\theta}}$  and  $C$  is independent of  $f \in W_{p_\theta}^{s_\theta}$ .

In particular, this means that if we have estimates

$$\|Tf\|_{L^\infty} \leq Ct^{d_0}\|f\|_{W_1^{N_0}}, \quad \|Tf\|_{L^2} \leq Ct^{d_1}\|f\|_{W_2^{N_1}},$$

then

$$\|Tf\|_{L^q} \leq C(1+t)^{d_p}\|f\|_{W_p^{N_p}}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ ,  $N_p = N_0(\frac{1}{p} - \frac{1}{q}) + \frac{2}{q}N_1$  and  $d_p = d_0(\frac{1}{p} - \frac{1}{q}) + \frac{2}{q}d_1$ . As usual, this reduces our task to finding  $L^1 - L^\infty$  and  $L^2 - L^2$  estimates in each case.

### 6.3 Estimates for large frequencies

Via the division of the integral above, it suffices to find  $L^p - L^q$  estimates for integrals of the form

$$\int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \widehat{f}(\xi) d\xi,$$

where  $a_j(\xi) = O(|\xi|^{-j})$  as  $|\xi| \rightarrow \infty$  is smooth (or is zero in a neighbourhood of 0), and  $\tau(\xi)$  is a complex-valued, smooth function which is  $O(|\xi|)$  as  $|\xi| \rightarrow \infty$  and  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$ . Note that  $\tau(\xi)$  does not have to be homogeneous.

By further judicious use of cut-off functions, we can split the considerations into the following cases of Theorem 2.18:

1.  $\tau(\xi)$  is separated from the real axis, i.e. there exists  $\delta > 0$  such that  $\text{Im } \tau(\xi) \geq \delta$  for all  $|\xi| \geq M$  (Theorem 2.1);
2.  $\tau(\xi)$  lies on the real axis (this case is contained in Theorems 2.3–2.12 since  $\tau$  is real valued);

Let us look at each of these in turn. We will not consider the case of  $\tau(\xi)$  tending asymptotically to the real axis as  $|\xi| \rightarrow \infty$  since it is not part of Theorem 2.18 and since we do not have at present any examples of such behaviour.

### 6.4 Phase separated from the real axis: Theorem 2.1

In this section, we consider the case where characteristic root  $\tau(\xi)$  is separated from the real axis for large  $|\xi|$ ; let us define  $\delta > 0$  to be a constant such that  $\text{Im } \tau(\xi) \geq \delta$  for all  $|\xi| \geq M$ . Again,  $\chi$  is a cut-off to the region (which may be unbounded) where these properties hold.

We claim that, for all  $t \geq 0$ , we have

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^\infty} &\leq Ce^{-\delta t} \|f\|_{W_1^{N_1+|\alpha|+r-j}}, \\ \left\| D_t^r D_x^\alpha \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^2} &\leq Ce^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}}, \end{aligned}$$

where  $N_1 > n$ ,  $r \geq 0$ ,  $\alpha$  multi-index. Indeed, these follow immediately from:

**Proposition 6.5.** Let  $\tau : U \rightarrow \mathbb{C}$  be a smooth function,  $U \subset \mathbb{R}^n$  open, and  $a_j = a_j(\xi) \in S_{1,0}^{-j}(U)$ . Assume:

- (i) there exists  $\delta > 0$  such that  $\text{Im } \tau(\xi) \geq \delta$  for all  $\xi \in U$ ;
- (ii)  $|\tau(\xi)| \leq C(1 + |\xi|)$  for all  $\xi \in U$ .

Then,

$$\left\| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right\|_{L^\infty(\mathbb{R}_x^n)} \leq C e^{-\delta t} \|f\|_{W_1^{N_1+|\alpha|+r-j}}$$

and

$$\left\| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} \leq C e^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}}$$

for all  $t \geq 0$ ,  $N_1 > n$ , multi-indices  $\alpha$ ,  $r \in \mathbb{R}$  and  $\widehat{f} \in C_0^\infty(U)$ .

Note that in the case of  $r = 0$ , condition (ii) may be omitted.

*Proof.* By the hypotheses on  $\tau(\xi)$  and  $a_j(\xi)$ , we can estimate

$$\begin{aligned} \left| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right| &\leq \int_U |e^{i\tau(\xi)t} a_j(\xi)| |\xi|^{|\alpha|} |\tau(\xi)|^r |\widehat{f}(\xi)| d\xi \\ &= \int_U e^{-\text{Im } \tau(\xi)t} |a_j(\xi)| |\xi|^{|\alpha|} |\tau(\xi)|^r |\widehat{f}(\xi)| d\xi \leq C e^{-\delta t} \int_U \langle \xi \rangle^{|\alpha|+r-j} |\widehat{f}(\xi)| d\xi \\ &\leq C e^{-\delta t} \int_U \langle \xi \rangle^{-N_1} d\xi \left\| \langle \xi \rangle^{N_1+|\alpha|+r-j} \widehat{f}(\xi) \right\|_{L^\infty} \leq C e^{-\delta t} \|f\|_{W_1^{N_1+|\alpha|+r-j}}. \end{aligned}$$

This proves the first inequality. For the second, note that Plancherel's theorem implies

$$\left\| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} = \|e^{i\tau(\xi)t} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\|_{L^2(U)};$$

then,

$$\begin{aligned} \int_U |e^{i\tau(\xi)t} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi)|^2 d\xi &\leq \int_U e^{-2\text{Im } \tau(\xi)t} |a_j(\xi)|^2 |\xi|^{2|\alpha|} |\tau(\xi)|^{2r} |\widehat{f}(\xi)|^2 d\xi \\ &\leq C e^{-2\delta t} \int_U \langle \xi \rangle^{2(|\alpha|+r-j)} |\widehat{f}(\xi)|^2 d\xi \leq C e^{-2\delta t} \|f\|_{W_2^{|\alpha|+r-j}}^2. \end{aligned}$$

This completes the proof of the proposition. □

We note that there may be different version of the  $L^\infty$ -estimate for the integral in Proposition 6.5. For example, applying Cauchy–Schwartz inequality to the estimate

$$\left| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right| \leq C e^{-\delta t} \int_U \langle \xi \rangle^{|\alpha|+r-j} |\widehat{f}(\xi)| d\xi$$

established in the proof, we get

$$\int_U \langle \xi \rangle^{|\alpha|+r-j} |\widehat{f}(\xi)| d\xi \leq \left( \int_U \langle \xi \rangle^{-2N'_1} d\xi \right)^{1/2} \left( \int_U \langle \xi \rangle^{2N'_1+2|\alpha|+2r-2j} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2},$$

from which we obtain the estimate

$$\left| \int_U e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right| \leq C e^{-\delta t} \|f\|_{W_2^{N'_1+|\alpha|+r-j}}, \quad (6.8)$$

with<sup>1</sup>  $N'_1 > \frac{n}{2}$ . Interpolating with the  $L^2$ -estimate from Proposition 6.5 yields estimate (2.7) in Section 2.1.

From Proposition 6.5, by the interpolation Theorem 6.4, we get

$$\left\| D_t^r D_x^\alpha \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{W_p^{N_p+|\alpha|+r-j}},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$ ,  $N_p \geq n(\frac{1}{p} - \frac{1}{q})$ ,  $r \geq 0$ ,  $\alpha$  a multi-index and  $f \in C_0^\infty(\mathbb{R}^n)$ . Thus, in this case we have exponential decay of the solution. This proves the first part of Theorem 2.1. The second part of the statement of Theorem 2.1 is a straightforward consequence.

## 6.5 Non-degenerate phase: Theorems 2.3 and 2.4

In this section, we will prove Theorems 2.3 and 2.4 and discuss the behavior of critical points of the phase. In fact, we will prove Theorem 2.3 since the proof of Theorem 2.4 can be given in the same way after restricting to a subset of variables on which the non-degenerate matrix  $A(\xi^0)$  is attained (possibly after a coordinate change). We will not write a further cut-off function  $\chi$  to a set  $U$  as in Theorems 2.3 and 2.4 to ensure that the results that we obtain are uniform over the positions of such sets  $U$ . However, we will keep in mind that we are only interested in the local in frequency region here, so all the integrals are convergent. So, we first consider the case where we have

$$\int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi,$$

and  $\det \text{Hess } \tau(\xi) \neq 0$  for all  $\xi \in \text{supp } a$ . Here we denote  $\tilde{x} = t^{-1}x$ . To estimate this, we first consider the oscillatory integral

$$\int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a(\xi) d\xi,$$

---

<sup>1</sup>Here  $N'_1$  does not have to be an integer.

where  $a = a(\xi) \in S_{1,0}^{-\mu}$ , some  $\mu \in \mathbb{R}$ ,  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$ , and, for some  $\xi^0 \in \mathbb{R}^n$ ,  $\tilde{x} + \nabla_\xi \tau(\xi^0) = 0$  and  $\det \text{Hess } \tau(\xi^0) \neq 0$ ; we refer to  $\xi^0$  as a (non-degenerate) critical point and we microlocalise around it. Let us assume that  $\xi^0$  is the only such critical point—if there are more than one, we use suitable cut-off functions to localise around each separately (we assume the set of critical points has no accumulation points). Indeed, let  $\vartheta \in C_0^\infty(\mathbb{R}^n)$  be supported in a neighbourhood  $V$  of  $\xi^0$  so that there are no other critical points in  $V$ . Then consider separately

$$\int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} a(\xi) \vartheta(\xi) d\xi \quad \text{and} \quad \int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} a(\xi) (1 - \vartheta)(\xi) d\xi.$$

The second integral, which we may assume contains no critical points in its support (otherwise introduce further cut-off functions around those), can be shown to decay faster than any power of  $t$ : note that away from the critical points, we can use the equality

$$e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} = \frac{\tilde{x} + \nabla \tau(\xi)}{it|\tilde{x} + \nabla \tau(\xi)|^2} \cdot \nabla_\xi [e^{i(\tilde{x} \cdot \xi + \tau(\xi))t}];$$

so, integrating by parts repeatedly shows that for any  $N \in \mathbb{N}$  sufficiently large,

$$\left| \int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} a(\xi) (1 - \vartheta)(\xi) d\xi \right| \leq C_N t^{-N}.$$

Let us return to the case when there is a critical point. We may assume that  $\text{Im } \tau(\xi^0) = 0$  since otherwise  $\text{Im } \tau(\xi^0) > 0$  in view of (2.2), and then Theorem 2.1 would actually give the exponential decay rate. We now claim that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} a(\xi) \vartheta(\xi) d\xi \right| &\leq C t^{-n/2} |\det \text{Hess}(\xi^0)|^{-1/2} |a(\xi^0) \chi(\xi^0)| \\ &\leq C t^{-n/2} |\det \text{Hess}(\xi^0)|^{-1/2} (1 + |\xi^0|)^{-\mu}. \end{aligned} \quad (6.9)$$

This is a consequence of the following theorem, see e.g. [Hör83a, Theorem 7.7.12, p. 228]:

**Theorem 6.6.** *Suppose  $\Phi = \Phi(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p)$  is a complex-valued smooth function in a neighbourhood of the origin  $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^p$  such that:*

- $\text{Im } \Phi \geq 0$ ;
- $\text{Im } \Phi(0, 0) = 0$ ;
- $\Phi'_x(0, 0) = 0$ ;
- $\det \Phi''_{xx}(0, 0) \neq 0$ .

*Also, suppose  $u \in C_0^\infty(K)$  where  $K$  is a small neighbourhood of  $(0, 0)$ . Then*

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} e^{i\omega \Phi(x, y)} u(x, y) dx - \right. \\ &\quad \left. ((\det(\omega \Phi''_{xx}/2\pi i))^0)^{-1/2} e^{i\omega \Phi^0} \sum_{j=0}^{N-1} (L_{\Phi, j} u)^0 \omega^{-j} \right| \leq C_N \omega^{-N-n/2}, \end{aligned}$$



for some choice of operators  $L_{\Phi,j}$ , where the notation  $G^0(y)$  (where  $G(x,y)$  is the function) means the function of  $y$  only which is in the same residue class modulo the ideal generated by  $\partial\Phi/\partial x_j$ ,  $j = 1, \dots, n$ .

The proof of this result uses the method of stationary phase; similar results (with slightly differing conditions and conclusions) can be found in [Sog93, (1.1.20), p. 49], [Ste93, Ch. VIII, 2.3, Proposition 6, p. 344], [Dui96, Proposition 1.2.4, p. 14] and [Trè80, p. 432, Ch. VIII, (2.15)–(2.16)], for example.

So, we have (6.9) as a simple consequence of this theorem; now, in order to show that

$$\left| \int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi))t} a(\xi) \vartheta(\xi) d\xi \right| \leq Ct^{-n/2}, \quad (6.10)$$

we must choose  $\mu \in \mathbb{R}$  suitably. In the sequel we may assume that  $M$  is even; if  $M$  is odd, the result follows by a standard interpolation argument taking the geometric mean.

Assume that  $|\det \text{Hess } \tau(\xi)| \geq C(1 + |\xi|)^{-M}$  for some  $M \in \mathbb{R}$ ; then taking  $\mu = M/2$ , we have this estimate. This extends the case of Klein–Gordon equation (which is done in [Hör97] pp.146–155) where  $\det \text{Hess } \tau(\xi) = (1 + |\xi|)^{-n-2}$ , so  $M = n + 2$ .

Let us now apply this result to our situation. We have

$$\int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \vartheta(\xi) \widehat{f}(\xi) d\xi,$$

where we may now think of  $\vartheta$  as  $\vartheta \in S_{1,0}^0$  to ensure uniformity, and  $a_j(\xi) = O(|\xi|^{-j})$  as  $|\xi| \rightarrow \infty$ ; we assume  $|\det \text{Hess } \tau(\xi)| \geq C(1 + |\xi|)^{-M}$ . Now, for each  $\nu \in \mathbb{N}$ , we have

$$\begin{aligned} a_j(\xi) &= (1 + |\xi|^2)^{-\nu} (1 + |\xi|^2)^\nu a_j(\xi) \\ &= \sum_{|\alpha| \leq \nu} c_\alpha (1 + |\xi|^2)^{-\nu} \xi^\alpha a_j(\xi) \xi^\alpha = \sum_{|\alpha| \leq \nu} a_{j,\alpha}(\xi) \xi^\alpha, \end{aligned}$$

where  $a_{j,\alpha}(\xi) = c_\alpha (1 + |\xi|^2)^{-\nu} \xi^\alpha a_j(\xi)$  is of order  $-j - 2\nu + |\alpha|$ . Moreover,  $a_{j,\alpha} \vartheta$  is of order  $-j - 2\nu + |\alpha|$  uniformly over  $\vartheta$  (satisfying the necessary uniform symbolic estimates). Taking  $\nu = M/2 - j$  and using that  $|\alpha| \leq \nu$ , we can ensure that the worst order of any of these symbols is  $-M/2$ . Then,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a_j(\xi) \vartheta(\xi) \widehat{f}(\xi) d\xi &= \sum_{|\alpha| \leq \nu} \int e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a_{j,\alpha}(\xi) \vartheta(\xi) \widehat{D^\alpha f}(\xi) d\xi \\ &= \sum_{|\alpha| \leq \nu} \left( \int e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a_{j,\alpha}(\xi) \vartheta(\xi) d\xi * D^\alpha f \right)(x). \end{aligned}$$

Then

$$\begin{aligned} &\left\| \sum_{|\alpha| \leq \nu} \int e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a_{j,\alpha}(\xi) \vartheta(\xi) d\xi * D^\alpha f(x) \right\|_{L^\infty} \\ &\leq \sum_{|\alpha| \leq \nu} \left\| \int e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a_{j,\alpha}(\xi) \vartheta(\xi) d\xi \right\|_{L^\infty} \|D^\alpha f\|_{L^1} \leq Ct^{-n/2} \|f\|_{W_1^{M/2-j}}, \end{aligned}$$

where we used estimate (6.10). Thus, we have an  $L^1 - L^\infty$  estimate in this case. To find an  $L^2 - L^2$  estimate is simpler: by the Plancherel's theorem, we have

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \vartheta(\xi) \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} &= C \|e^{i\tau(\xi)t} a_j(\xi) \vartheta(\xi) \widehat{f}(\xi)\|_{L^2(\mathbb{R}_\xi^n)} \\ &\leq C \|\langle \xi \rangle^{-j} \widehat{f}(\xi)\|_{L^2} \leq C \|f\|_{W_2^{-j}}. \end{aligned}$$

Using the interpolation Theorem 6.4 and noting that all integrals are bounded for small  $t$ , we obtain Theorem 2.3.

**Behaviour of Critical Points:** Above, we assumed that  $\xi^0$  was the only critical point of the phase function; this is not such an unreasonable assumption as the following observation shows:

**Lemma 6.7.** *If the matrix of second order derivatives  $\text{Hess } \tau(\xi)$  is positive definite for all  $\xi$ , then the integral*

$$\int_{\mathbb{R}^n} e^{i(\tilde{x} \cdot \xi + \tau(\xi)t)} a(\xi) d\xi$$

*has only one critical point.*

*Proof.* Suppose  $\xi^1, \xi^2 \in \mathbb{R}^n$  are two such critical points. So  $\tilde{x} + \nabla_\xi \tau(\xi^1) = \tilde{x} + \nabla_\xi \tau(\xi^2)$ , or  $\partial_{\xi_j} \tau(\xi^1) = \partial_{\xi_j} \tau(\xi^2)$  for each  $j = 1, \dots, n$ . Thus, by the fundamental theorem of calculus, for all  $j = 1, \dots, n$ , we have

$$0 = \partial_{\xi_j} \tau(\xi^1) - \partial_{\xi_j} \tau(\xi^2) = \int_0^1 (\xi^1 - \xi^2) \cdot \nabla_\xi (\partial_{\xi_j} \tau)(\xi^1 + s(\xi^2 - \xi^1)) ds.$$

But this means that  $(\xi^1 - \xi^2) \text{Hess } \tau(\xi^1 + s(\xi^2 - \xi^1))(\xi^1 - \xi^2) = 0$  for all  $s$  since the Hessian is positive definite; and since it is never zero, we have that  $\xi^1 - \xi^2 = 0$ , which shows that there is at most one critical point.  $\square$

An example of such an operator is the Klein–Gordon equation.

**Remark 6.8.** *In general, another consequence of  $\text{Hess } \tau(\xi)$  being positive definite is that the level sets  $S_\lambda = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$ ,  $\lambda \in \mathbb{R}$  are all strictly convex; indeed, if we take a smooth curve  $\xi(s) \in S_\lambda$ ,  $s \geq 0$ , where  $\xi(0) = \xi^0$  and, by assumption,  $\dot{\xi}(s) \neq 0$ , then  $\nabla \tau(\xi(s)) \cdot \dot{\xi}(s) = 0$  (differentiate  $\tau(\xi(s)) = \lambda$ ), and (differentiating again)*

$$\dot{\xi}(s)^T \cdot \text{Hess } \tau(\xi(s)) \cdot \dot{\xi}(s) + \nabla \tau(\xi(s)) \cdot \ddot{\xi}(s) = 0.$$

*Then, since  $\text{Hess } \tau(\xi)$  is positive definite, the first term in this sum is positive, hence the second is negative—which means that the angle between  $\nabla \tau(\xi(s))$ , that is, the normal to the level set, and  $\dot{\xi}(s)$  is strictly greater than  $\pi/2$ , so the level set is strictly convex. In particular, this shows that imposing the condition  $\text{Hess } \tau(\xi)$  positive definite is stronger than imposing the convexity condition of Definition 2.5, and making it clear why we get a faster rate of decay in this case (see the next section for that case).*

**Remark 6.9.** *If  $\text{rank Hess } \tau(\xi) = n - 1$ , then a similar argument can be used to prove the corresponding part of Theorem 2.18, i.e. that there is decay of order  $-\frac{n-1}{2}$ . This is a consequence of an extension to Theorem 6.6—see Hörmander [Hör83a, Section 7.7].*

## 6.6 Phase satisfies the convexity condition: Theorem 2.8

The case of real roots and real-valued phase functions subdivides into the following subcases, each of which yields a different decay rate:

- (i)  $\det \text{Hess } \tau(\xi) \neq 0$ ; in this case we use the method of stationary phase in the same way as in Section 6.5, with same result;
- (ii)  $\det \text{Hess } \tau(\xi) = 0$  and  $\tau(\xi)$  satisfies the convexity condition of Definition 2.5; in this case we use Theorem 4.8;
- (iii) the general case when  $\det \text{Hess } \tau(\xi) = 0$  (i. e.  $\tau(\xi)$  does not satisfy the convexity condition); in this case, we use Theorem 5.3.

We assume throughout that  $\tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$  or  $\tau(\xi) \leq 0$  for all  $\xi \in \mathbb{R}^n$ . This is valid because for the characteristic roots lying on the real axis, there exists a linear function  $\tilde{\tau}(\xi)$  such that  $\tilde{\tau}_k(\xi) := \tau_k(\xi) - \tilde{\tau}(\xi)$  is either everywhere non-negative or everywhere non-positive, and, if  $\tau_k(\xi)$  satisfies the convexity condition, so does  $\tilde{\tau}_k(\xi)$ . A proof for this in the case of homogeneous symbols is given in [Sug94] and we recall this result here for completeness:

**Proposition 6.10.** *Let  $\varphi_k(\xi)$ ,  $k = 1, \dots, m$ , be the characteristic roots of a strictly hyperbolic operator with homogeneous symbol of order  $m$ , ordered as  $\varphi_1(\xi) > \varphi_2(\xi) > \dots > \varphi_m(\xi)$  for  $\xi \neq 0$ . Suppose that all the Hessians  $\varphi_k''(\xi)$  are semi-definite for  $\xi \neq 0$ . Then there exists a polynomial  $\alpha(\xi)$  of order one such that  $\varphi_{m/2}(\xi) > \alpha(\xi) > \varphi_{m/2+1}$  (if  $m$  is even) or  $\alpha(\xi) = \varphi_{(m+1)/2}(\xi)$  (if  $m$  is odd). Moreover, the hypersurfaces  $\Sigma_k = \{\xi \in \mathbb{R}^n; \tilde{\varphi}_k = \pm 1\}$  with  $\tilde{\varphi}_k(\xi) = \varphi_k(\xi) - \alpha(\xi)$  ( $k \neq (m+1)/2$ ) are convex and  $\gamma(\Sigma_k) \leq 2[m/2]$ .*

The generalisation of this proposition to the case of non-homogeneous symbols follows using the perturbation results in Section 3.

Assume that  $\tau(\xi)$  satisfies the convexity condition of Definition 2.5. Set  $\gamma \equiv \gamma(\tau) := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau))$ , where, as before,

$$\Sigma_\lambda(\tau) = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} .$$

and

$$\gamma(\Sigma_\lambda(\tau)) := \sup_{\sigma \in \Sigma_\lambda(\tau)} \sup_P \gamma(\Sigma_\lambda(\tau); \sigma, P)$$

where the second supremum is over planes  $P$  containing the normal to  $\Sigma_\lambda(\tau)$  at  $\sigma$  and  $\gamma(\Sigma_\lambda(\tau); \sigma, P)$  denotes the order of the contact between the line  $T_\sigma \cap P$ — $T_\sigma$  is the tangent plane at  $\sigma$ —and the curve  $\Sigma_\lambda(\tau) \cap P$ .

We have the following results which ensures that this is finite:

**Lemma 6.11.** *Suppose  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is a characteristic root of a linear  $m^{\text{th}}$  order constant coefficient strictly hyperbolic partial differential operator. Then, there exists a homogeneous function of order 1,  $\varphi(\xi)$ , a characteristic root of the principal symbol, such that*

$$\gamma(\Sigma_\lambda(\tau)) \rightarrow \gamma(\Sigma_1(\varphi)) \text{ as } \lambda \rightarrow \infty.$$

*If we assume that  $\gamma(\Sigma_\lambda(\tau)) < \infty$  for all  $\lambda > 0$ , then we have  $\gamma(\tau) < \infty$ .*

*Proof.* This is true because:

- (a) by Proposition 3.5, Part II,  $\Sigma_\lambda(\tau)$  is near to  $\Sigma_\lambda(\varphi)$  for large  $\lambda$  in a suitable metric;
- (b) by the homogeneity of  $\varphi$ , if  $|\lambda - \lambda'|$  is sufficiently small, then  $\Sigma_\lambda(\varphi)$  is near to  $\Sigma_{\lambda'}(\varphi)$  for large  $\lambda$  in the same metric;
- (c) Proposition 3.5, Part IV, ensures that  $T_\sigma(\tau)$  is near to  $T_\sigma(\varphi)$  (because derivatives of  $\tau$  tend to those of  $\varphi$ ) for large  $\lambda$ ;
- (d) so, with  $\Sigma_\lambda(\tau)$  and  $T_\sigma(\tau)$  near to (in a suitable sense) the corresponding data of  $\varphi$  for large  $\lambda$ , it is clear that the  $\gamma(\Sigma_\lambda(\tau); \sigma, P)$  is near to  $\gamma(\Sigma_\lambda(\varphi); \sigma, P)$ , and hence  $\gamma(\Sigma_\lambda(\tau))$  is near to  $\gamma(\Sigma_\lambda(\varphi))$ ;
- (e) finally,  $\gamma(\Sigma_1(\varphi)) = \gamma(\Sigma_\lambda(\varphi))$  by homogeneity.

□

In order to prove Theorem 2.8, we shall show that if  $a_j \in S_{1,0}^{-j}$  is a symbol of order  $-j$ , then we have the estimate

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q} \leq C(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_{p,j}}}, \quad (6.11)$$

for all  $t \geq 0$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < p \leq 2$ , and  $f \in C_0^\infty(\mathbb{R}^n)$ . The Sobolev order  $N_{p,j}$  (which does not have to be an integer here) is worse for small times, being  $N_{p,j} \geq n(\frac{1}{p} - \frac{1}{q}) - j$ . It can be actually improved for large times, which will be done in estimate (6.16).

**Besov Space Reduction:** We begin by following Brenner [Bre75] and also Sugimoto [Sug94] in using the theory of Besov spaces and Paley decomposition to reduce this to showing, for all  $t \geq 0$ , the estimate

$$\left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi)) \right\|_{L^q} \leq C(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{N_{p,j}}}; \quad (6.12)$$

here  $\{\Phi_l(\xi)\}_{l=0}^\infty$  is a Hardy–Littlewood partition: let  $\Phi \in C_0^\infty(\mathbb{R}^n)$  be such that

$$\text{supp } \Phi = \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad \Phi(\xi) > 0 \text{ for } \frac{1}{2} < |\xi| < 2,$$

$$\text{and } \sum_{k=-\infty}^{\infty} \Phi(2^{-k}\xi) = 1 \text{ for } \xi \neq 0,$$

and set

$$\Phi_0(\xi) = 1 - \sum_{l=1}^{\infty} \Phi(2^{-l}\xi), \quad \Phi_l(\xi) := \Phi(2^{-l}\xi), \quad l \in \mathbb{N}.$$

Now, recall the definition of a Besov space, as given in, for example, Bergh and L fstr m [BL76]:

**Definition 6.12.** *For suitable  $p, q, s \in \mathbb{R}$  define the Besov norm by*

$$\|f\|_{B_{p,q}^s} := \|\mathcal{F}^{-1}(\Phi_0(\xi)\widehat{f}(\xi))\|_{L^p} + \left( \sum_{l=1}^{\infty} (2^{sl} \|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{f}(\xi))\|_{L^p})^p \right)^{1/q};$$

the Besov space  $B_{p,q}^s$  is the space of functions in  $\mathcal{S}'(\mathbb{R}^n)$  for which this norm is finite.

This result is the main one we shall need:

**Theorem 6.13** ([BL76], Theorem 6.4.4). *The following inclusions hold:*

$$B_{p,p}^s \subset W_p^s \subset B_{p,2}^s \quad \text{and} \quad B_{q,2}^s \subset W_q^s \subset B_{q,q}^s$$

for all  $s \in \mathbb{R}$ ,  $1 < p \leq 2$ ,  $2 \leq q < \infty$ .

There are some weaker versions of these embeddings for  $p = 1$ . Using this theorem, we have

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}^n)} &= (2\pi)^n \left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \widehat{f}(\xi))(t, x) \right\|_{L^q} \\ &\leq C \left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \widehat{f}(\xi)) \right\|_{B_{q,2}^0} \\ &= C \left( \sum_{l=0}^{\infty} \left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi)) \right\|_{L^q}^2 \right)^{1/2} \\ &= C \left( \sum_{l=0}^{\infty} \left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \sum_{r=l-1}^{l+1} \Phi_r(\xi) \widehat{f}(\xi)) \right\|_{L^q}^2 \right)^{1/2}; \end{aligned}$$

in the final line we have used that  $\sum_{r=l-1}^{l+1} \Phi_r(\xi) = 1$  on  $\text{supp } \Phi_l(\xi)$  by the structure of the partition of unity. Now, assuming that (6.12) holds, this can be further estimated:

$$\begin{aligned} &\left( \sum_{l=0}^{\infty} \left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \sum_{r=l-1}^{l+1} \Phi_r(\xi) \widehat{f}(\xi)) \right\|_{L^q}^2 \right)^{1/2} \\ &\leq C t^{-\frac{n-1}{\gamma} \left( \frac{1}{p} - \frac{1}{q} \right)} \left( \sum_{l=0}^{\infty} \left( \sum_{r=l-1}^{l+1} \left\| \mathcal{F}^{-1}(\Phi_r(\xi) \widehat{f}(\xi)) \right\|_{W_p^{N_{p,j}}} \right)^2 \right)^{1/2} \\ &\leq C t^{-\frac{n-1}{\gamma} \left( \frac{1}{p} - \frac{1}{q} \right)} \left( \sum_{l=0}^{\infty} \sum_{r=l-1}^{l+1} \left\| \mathcal{F}^{-1}(\Phi_r(\xi) \widehat{f}(\xi)) \right\|_{W_p^{N_{p,j}}}^2 \right)^{1/2} \\ &\leq C t^{-\frac{n-1}{\gamma} \left( \frac{1}{p} - \frac{1}{q} \right)} \left( \sum_{l=0}^{\infty} \left\| \mathcal{F}^{-1}(\Phi_l(\xi) \widehat{f}(\xi)) \right\|_{W_p^{N_{p,j}}}^2 \right)^{1/2}. \end{aligned}$$

Finally, using Theorem 6.13 once again, we get

$$\begin{aligned}
\left( \sum_{l=0}^{\infty} \|\mathcal{F}^{-1}(\Phi_l(\xi) \widehat{f}(\xi))\|_{W_p^{N_{p,j}}}^2 \right)^{\frac{1}{2}} &\leq C \left( \sum_{l=0}^{\infty} \sum_{|\alpha| \leq N_{p,j}} \|D_x^\alpha [\mathcal{F}^{-1}(\Phi_l(\xi) \widehat{f}(\xi))]\|_{L^p}^2 \right)^{\frac{1}{2}} \\
&= C \sum_{|\alpha| \leq N_{p,j}} \left( \sum_{l=0}^{\infty} \|\mathcal{F}^{-1}(\Phi_l(\xi) \widehat{D^\alpha f}(\xi))\|_{L^p}^2 \right)^{1/2} \\
&= C \sum_{|\alpha| \leq N_{p,j}} \|D^\alpha f\|_{B_{p,2}^0} \leq C \|f\|_{W_p^{N_{p,j}}} .
\end{aligned}$$

Combining these estimates shows that (6.12) implies (6.11) as desired. So, it suffices to prove (6.12); moreover, as shown above, this requires us to show two estimates and then interpolate—Theorem 6.4 yields:

$$\|\mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi))(t, x)\|_{L^\infty} \leq C(1+t)^{-\frac{n-1}{\gamma}} \|f\|_{W_1^{N_1-j}} , \quad (6.13)$$

$$\|\mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi))(t, x)\|_{L^2} \leq C \|f\|_{W_2^{-j}} , \quad (6.14)$$

where  $N_1 > n$ .

**$L^2 - L^2$  estimate:** Since  $\tau(\xi)$  is real-valued and  $a_j(\xi) = O(|\xi|^{-j})$  as  $|\xi| \rightarrow \infty$ , by Plancherel's theorem we get

$$\begin{aligned}
\|\mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi))\|_{L^2} &= \int_{\mathbb{R}^n} |e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi)|^2 d\xi \\
&\leq C \int_{|\xi| \geq M} |\xi|^{-2j} |\widehat{f}(\xi)|^2 d\xi \leq C \|f\|_{W_2^{-j}} .
\end{aligned}$$

Note that  $C$  is independent of  $l$  because  $a_j(\xi)|\xi|^j$  is uniformly bounded in  $\mathbb{R}^n$ . This proves the required estimate (6.14).

**$L^1 - L^\infty$  estimate:** First, suppose  $0 \leq t < 1$ ; then

$$\begin{aligned}
\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi) d\xi \right\|_{L^\infty} &\leq C \int_{|\xi| \geq M} |\xi|^{-j} |\widehat{f}(\xi)| d\xi \\
&\leq C \int_{|\xi| \geq M} |\xi|^{-N_1} d\xi \|\langle \xi \rangle^{N_1-j} \widehat{f}(\xi)\|_{L^\infty} \\
&\leq C \|f\|_{W_1^{N_1-j}} ,
\end{aligned} \quad (6.15)$$

where  $N_1 > n$ .

For  $t \geq 1$ , we show

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi) d\xi \right\|_{L^\infty} \leq C t^{-\frac{n-1}{\gamma}} \|f\|_{W_1^{n-\frac{n-1}{\gamma}-j}} . \quad (6.16)$$

Together (6.15) and (6.16) will imply (6.13). We claim now that it suffices to prove that there exists a constant  $C > 0$  which is independent of  $l$  such that, for all  $t \geq 1$ ,

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \langle \xi \rangle^{\frac{n-1}{\gamma} - n + j} \Phi_l(\xi) d\xi \right\|_{L^\infty} \leq C t^{-\frac{n-1}{\gamma}}. \quad (6.17)$$

Indeed,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi) d\xi &= (2\pi)^n \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi)) \\ &= (2\pi)^n \mathcal{F}_{\xi \rightarrow x}^{-1}[e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi)] * f(x) \\ &= \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) d\xi \right) * f(x), \end{aligned}$$

and, by the definition of the symbol of  $\langle D_x \rangle$ , we have

$$\begin{aligned} &\left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) d\xi \right) * f(x) \\ &= \left( \int_{\mathbb{R}^n} \langle D_x \rangle^{n - \frac{n-1}{\gamma} - j} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \langle \xi \rangle^{\frac{n-1}{\gamma} - n + j} d\xi \right) * f(x) \\ &= \langle D_x \rangle^{n - \frac{n-1}{\gamma} - j} \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \langle \xi \rangle^{\frac{n-1}{\gamma} - n + j} d\xi \right) * f(x) \\ &= \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \Phi_l(\xi) \langle \xi \rangle^{\frac{n-1}{\gamma} - n + j} d\xi \right) * \langle D_x \rangle^{n - \frac{n-1}{\gamma} - j} f(x); \end{aligned}$$

also,

$$\|g * h\|_{L^\infty} \leq \|g\|_{L^\infty} \|h\|_{L^1},$$

for all  $g \in L^\infty(\mathbb{R}^n)$ ,  $h \in L^1(\mathbb{R}^n)$ . Combining all these shows that (6.17) implies (6.16).

In order to prove (6.17), we can use Theorem 4.8 as  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is assumed to satisfy the convexity condition; let us check that each hypothesis holds. In addition to properties ensured by Proposition 3.8, we have:

- Property (i) suffices for the hypothesis (i) of Theorem 4.8 to hold since  $a_j(\xi)$  is supported away from the origin.
- $a_j(\xi) \langle \xi \rangle^{\frac{n-1}{\gamma} - n + j}$  is a symbol of order  $\frac{n-1}{\gamma} - n$  since  $a \in S^{-j}$  and because it is zero in a neighbourhood of the origin.
- the partition of unity  $\{\Phi_l(\xi)\}_{l=1}^\infty$  is in the form of  $g_R(\xi)$  as required by Theorem 4.8.

Also,  $\gamma < \infty$  by Lemma 6.11 above. Therefore, for  $t \geq 1$ , we get

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) |\xi|^{\frac{n-1}{\gamma} - n + j} \Phi_l(\xi) d\xi \right| \leq C t^{-\frac{n-1}{\gamma}}.$$

Hence, we have (6.16), which, together with (6.15), proves (6.13); this completes the proof of Theorem 2.8 on real axis with convexity condition  $\gamma$ .

## 6.7 Results without convexity: Theorem 2.12

The general case depends upon Theorem 5.3, just as the case where the convexity condition holds depends upon Theorem 4.8. Here we assume that  $\tau$  is real valued. We introduce  $\gamma_0 \equiv \gamma_0(\tau) := \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\tau))$ , where,

$$\gamma_0(\Sigma_\lambda(\tau)) := \sup_{\sigma \in \Sigma_\lambda(\tau)} \inf_P \gamma(\Sigma_\lambda(\tau); \sigma, P)$$

(all notation as before). For this quantity we have the analogous result to Lemma 6.11, which can be proved in the same way:

**Lemma 6.14.** *If  $\tau : \mathbb{R}^n \rightarrow \mathbb{R}$  is a characteristic root of a linear  $m^{\text{th}}$  order constant coefficient strictly hyperbolic partial differential operator, then, there exists a homogeneous function of order 1,  $\varphi(\xi)$ , a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\tau)) \rightarrow \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda \rightarrow \infty.$$

If we assume that  $\gamma_0(\Sigma_\lambda(\tau)) < \infty$  for all  $\lambda > 0$ , then we have  $\gamma_0(\tau) < \infty$ .

We shall show

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q} \leq C(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p} - \frac{1}{q})} \|f\|_{W_p^{N_{p,j}}},$$

for all  $t \geq 0$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ ,  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $N_{p,j} \geq n(\frac{1}{p} - \frac{1}{q}) - j$  and  $N_{1,j} > n - j$ . Similarly to (6.16), the Sobolev order  $N_{p,j}$  can be improved for large times.

As in the case of Section 6.6, this can be reduced, via a Besov space reduction the interpolation Theorem 6.4, to showing

$$\begin{aligned} \|\mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi))(t, x)\|_{L^\infty} &\leq C(1+t)^{-\frac{1}{\gamma_0}} \|f\|_{W_1^{N_1-j}}, \\ \|\mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi) \Phi_l(\xi) \widehat{f}(\xi))(t, x)\|_{L^2} &\leq C \|f\|_{W_2^{-j}}, \end{aligned}$$

where the partition of unity  $\{\Phi_l(\xi)\}_{l=1}^\infty$  is as above and  $N_1 > n$ .

The  $L^2$  estimate follows by the Plancherel's theorem in the same way as before.

For the  $L^1 - L^\infty$  estimate, the case  $0 \leq t < 1$  is as in (6.15); for  $t \geq 1$  it suffices to show (see the earlier argument),

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi)t)} a_j(\xi) \langle \xi \rangle^{\frac{1}{\gamma_0} - n + j} \Phi_l(\xi) d\xi \right\|_{L^\infty} \leq C t^{-1/\gamma_0}.$$

This follows by Theorem 5.3: the hypotheses of this hold by the same arguments as above (see Proposition 3.8)—the convexity condition is not required for the perturbation methods employed—and Lemma 6.14. This completes the proof of 2.12.



## 6.8 Asymptotic properties of complex phase functions

Here we consider what happens when the phase function  $\tau(\xi)$  is complex valued and look at its behaviour for large frequencies. In particular, this is related to the case

$$\operatorname{Im} \tau(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty.$$

Unlike in the case of the phase function  $\tau(\xi)$  lying on the real axis, here we do not consider a case where the phase function satisfies a “convexity condition”. The reason for this is twofold: firstly, there is no straightforward analog of the convexity condition for real-valued phase functions as the presence of the non-zero imaginary part causes problems; secondly, there are no common examples of this situation, and hence it does not seem worthwhile developing a complicated theory for this situation.

If  $\det \operatorname{Hess} \tau(\xi) \neq 0$ , the analysis can be done in exactly the same way as that in Section 6.5, since Theorem 6.6 holds for integrals with complex phase functions.

In general, we can derive certain properties of real and imaginary parts of  $\tau(\xi)$  using perturbation arguments of Section 3. For example, for the index  $\gamma_0 = \gamma_0(\operatorname{Re} \tau) = \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\operatorname{Re} \tau))$  we can note the following:

**Lemma 6.15.** *If  $\tau : \mathbb{R}^n \rightarrow \mathbb{C}$  is a characteristic root of a linear  $m^{\text{th}}$  order constant coefficient strictly hyperbolic partial differential operator such that  $\operatorname{Im} \tau(\xi) \rightarrow 0$  as  $|\xi| \rightarrow \infty$ , then, there exists a homogeneous function of order 1,  $\varphi(\xi)$ , a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\operatorname{Re} \tau)) \rightarrow \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda \rightarrow \infty.$$

*In particular,  $\gamma_0(\operatorname{Re} \tau) < \infty$ .*

*Proof.* The hypothesis that the imaginary part goes to zero as  $|\xi| \rightarrow \infty$  implies that  $|\tau(\xi) - \operatorname{Re} \tau(\xi)| \rightarrow 0$  as  $|\xi| \rightarrow \infty$ . With this additional observation, the proof of Lemma 6.11 can then be used once more.  $\square$

In addition to Proposition 3.8, we will now prove the following refined perturbation properties:

**Proposition 6.16.** *Suppose  $\tau : \mathbb{R}^n \rightarrow \mathbb{C}$  is a characteristic root of the strictly hyperbolic Cauchy problem (1.1). Assume that it is a smooth function satisfying  $\operatorname{Im} \tau(\xi) \geq 0$ . Assume also that the roots  $\phi_k(\xi)$ ,  $k = 1, \dots, m$ , of the principal part  $L_m$  are non-zero for all  $\xi \neq 0$ . Then we have the following properties:*

- (i) *for all multi-indices  $\alpha$  there exist constants  $M, C_\alpha, C'_\alpha > 0$  such that*

$$|\partial_\xi^\alpha \operatorname{Re} \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|}$$

*and*

$$|\partial_\xi^\alpha \operatorname{Im} \tau(\xi)| \leq C'_\alpha (1 + |\xi|)^{-|\alpha|};$$

*for all  $|\xi| \geq M$ .*

- (ii) *there exist constants  $M, C > 0$  such that for all  $|\xi| \geq M$  we have  $|\operatorname{Re} \tau(\xi)| \geq C|\xi|$ ;*
- (iii) *there exists a constant  $C_0 > 0$  such that  $|\partial_\omega \operatorname{Re} \tau(\lambda\omega)| \geq C_0$  for all  $\omega \in \mathbb{S}^{n-1}$  and sufficiently large  $\lambda > 0$ ;*
- (iv) *there exists a constant  $R_1 > 0$  such that, for all sufficiently large  $\lambda > 0$ ,*

$$\frac{1}{\lambda} \{ \xi \in \mathbb{R}^n : \operatorname{Re} \tau(\xi) = \lambda \} \subset B_{R_1}(0).$$

*Proof.*(i) The statements follow by Proposition 3.5: Part III implies that for all  $|\xi| \geq N$  and multi-indices  $\alpha$ ,

$$|\partial_\xi^\alpha \operatorname{Re} \tau(\xi)| \leq |\partial_\xi^\alpha \tau(\xi)| \leq C|\xi|^{1-|\alpha|},$$

which suffices for the first part of (i). Furthermore, Part IV tells us that for all  $|\xi| \geq N$  and multi-indices  $\alpha$ ,

$$|\partial_\xi^\alpha [\operatorname{Re} \tau(\xi) - \varphi(\xi)] + i\partial_\xi^\alpha \operatorname{Im} \tau(\xi)| = |\partial_\xi^\alpha \tau(\xi) - \partial_\xi^\alpha \varphi(\xi)| \leq C|\xi|^{-|\alpha|},$$

where  $\varphi(\xi)$  is a characteristic root of the principal part (and is thus real-valued by definition of hyperbolicity); this implies that, for all  $|\xi| \geq N$  and multi-indices  $\alpha$ ,

$$|\partial_\xi^\alpha [\operatorname{Re} \tau(\xi) - \varphi(\xi)]| \leq C|\xi|^{-|\alpha|} \text{ and } |\partial_\xi^\alpha \operatorname{Im} \tau(\xi)| \leq C|\xi|^{-|\alpha|}. \quad (6.18)$$

The second of these gives us the second part of (i).

- (ii) We note that there exist constants  $C, C', C'', M > 0$  such that, for all  $|\xi| \geq M$ ,

$$|\operatorname{Re} \tau(\xi)| \geq |\tau(\xi)| - |\operatorname{Im} \tau(\xi)| \geq C'|\xi| - C'' \geq C|\xi|.$$

Here we have used (3.19), which did not require  $\tau$  to be real-valued (nor to satisfy the convexity condition), simply to be a characteristic root of a linear constant coefficient strictly hyperbolic partial differential equation, and the second part of (6.18).

- (iii) This follows in a similar way: using (6.18), we have, for  $\lambda \geq M$ , some  $M > 0$ , that

$$|\partial_\omega \operatorname{Re} \tau(\lambda\omega)| \geq |\partial_\omega \tau(\lambda\omega)| - |\partial_\omega \operatorname{Im} \tau(\lambda\omega)| \geq C' - C''\lambda^{-1} \geq C.$$

- (iv) This follows from  $|\operatorname{Re} \tau(\xi) - \varphi(\xi)| \leq C$  for all  $\xi \in \mathbb{R}^n$  which holds in all  $\mathbb{R}^n$  by Part II of Proposition 3.5.

□

## 6.9 Estimates for bounded frequencies away from multiplicities

In the following sections we find  $L^p - L^q$  estimates for integrals of the kind

$$\int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi,$$

where  $\Omega \subset \mathbb{R}^n$  is open and bounded,  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $a \in C_0^\infty(\Omega)$ ,  $\tau \in C^\infty(\Omega)$  and  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in \Omega$ .

As in the case of large  $|\xi|$ , we can further split this into three main cases by using suitable cut-off functions:

1.  $\tau(\xi)$  is separated from the real axis for all  $\xi \in \Omega$  (Theorem 2.1);
2.  $\tau(\xi)$  meets the real axis with order  $s < \infty$  at a point  $\xi^0 \in \Omega$  (Theorem 2.16);
3.  $\tau(\xi)$  lies on the real axis for all  $\xi \in \Omega$ .

We look at each in turn.

## 6.10 Phase separated from the real axis: Theorem 2.1 again

Similarly to the case for large  $|\xi|$ , we show that when the phase function  $\tau(\xi)$  is separated from the real axis (here, for  $\xi \in \Omega$ ,  $\Omega$  is a bounded set),

$$\left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{L^p}, \quad (6.19)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ ,  $r \geq 0$ ,  $\alpha$  a multi-index,  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $\delta > 0$  is a constant such that  $\text{Im } \tau(\xi) \geq \delta$  for all  $\xi \in \Omega$  and  $C \equiv C_{\Omega, r, \alpha, p} > 0$ . So, in this case we also have exponential decay of the solution.

By interpolating (Theorem 6.4), it suffices to show for such  $\tau(\xi)$

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^\infty} &\leq C e^{-\delta t} \|f\|_{L^1}, \\ \left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^2} &\leq C e^{-\delta t} \|f\|_{L^2}, \end{aligned}$$

for  $t \geq 0$ , where  $r \geq 0$  and  $\alpha$  is a multi-index.

These are proved in a similar way to Proposition 6.5, but noting that the boundedness of  $\Omega$  and the continuity in  $\Omega$  of  $\tau(\xi)^r a(\xi)$  ensure there exists a constant  $C_{\Omega, r, \alpha} \equiv C > 0$  such that  $|\tau(\xi)|^r |a(\xi)| |\xi|^{|\alpha|} \leq C$  for all  $\xi \in \Omega$ . Then, for all  $t \geq 0$  and  $r, \alpha$  as above, we can estimate

$$\begin{aligned} \left| D_t^r D_x^\alpha \left( \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right) \right| &= \left| \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) d\xi \right| \\ &\leq C \int_{\Omega} e^{-\text{Im } \tau(\xi)t} |a(\xi)| |\xi|^{|\alpha|} |\tau(\xi)|^r |\widehat{f}(\xi)| d\xi \\ &\leq C \int_{\Omega} e^{-\text{Im } \tau(\xi)t} |\widehat{f}(\xi)| d\xi \leq C e^{-\delta t} \|\widehat{f}\|_{L^\infty(\Omega)} \leq C e^{-\delta t} \|f\|_{L^1}, \end{aligned}$$

and

$$\begin{aligned}
\left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^2(\mathbb{R}^n)} &= \left\| e^{i\tau(\xi)t} a(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi) \right\|_{L^2(\Omega)} \\
&= \left( \int_{\Omega} e^{-2\operatorname{Im} \tau(\xi)t} |a(\xi)|^2 |\xi^\alpha|^2 |\tau(\xi)|^{2r} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \\
&\leq C e^{-\delta t} \|\widehat{f}\|_{L^2(\Omega)} \leq C e^{-\delta t} \|f\|_{L^2}.
\end{aligned}$$

We have now completed the proof of Theorem 2.1.

## 6.11 Roots meeting the real axis: Theorem 2.16

In the case of bounded  $|\xi|$ , we must also consider the situation where the phase function  $\tau(\xi)$  meets the real axis. Suppose  $\xi^0 \in \Omega$  is such a point, i.e.  $\operatorname{Im} \tau(\xi^0) = 0$ , while in each punctured ball around  $\xi^0$ ,  $B'_\varepsilon(\xi^0) \subset \Omega$ ,  $\varepsilon > 0$ , we have  $\operatorname{Im} \tau(\xi) > 0$ . Then,  $\xi^0$  is a root of  $\operatorname{Im} \tau(\xi)$  of some finite order  $s$ : indeed, if  $\xi^0$  were a zero of  $\operatorname{Im} \tau(\xi)$  of infinite order, then, by the analyticity of  $\operatorname{Im} \tau(\xi)$  at  $\xi^0$  (which follows straight from the analyticity of  $\tau(\xi)$  at  $\xi^0$ ) it would be identically zero in a neighbourhood of  $\xi^0$ , contradicting the assumption.

Furthermore, we claim that  $s \geq 2$ ,  $s$  is even, and that there exist constants  $c_0, c_1 > 0$  such that, for all  $\xi$  sufficiently close to  $\xi^0$ , we have

$$c_0 |\xi - \xi^0|^s \leq |\operatorname{Im} \tau(\xi)| \leq c_1 |\xi - \xi^0|^2.$$

Indeed, the Taylor expansion of  $\operatorname{Im} \tau(\xi)$  around  $\xi^0$ ,

$$\operatorname{Im} \tau(\xi) = \sum_{i=1}^n \partial_{\xi_i} \operatorname{Im} \tau(\xi^0) (\xi_i - (\xi^0)_i) + O(|\xi - \xi^0|^2),$$

is valid for  $\xi \in B_\varepsilon(\xi^0) \subset \Omega$  for some small  $\varepsilon > 0$ . Now, if  $\xi \in B_\varepsilon(\xi^0)$ , then  $-\xi + 2\xi^0 \in B_\varepsilon(\xi^0)$  also. However,

$$\operatorname{Im} \tau(-\xi + 2\xi^0) = - \sum_{i=1}^n \partial_{\xi_i} \operatorname{Im} \tau(\xi^0) (\xi_i - (\xi^0)_i) + O(|\xi - \xi^0|^2);$$

thus, for  $\varepsilon > 0$  chosen small enough, this means that either  $\operatorname{Im} \tau(\xi) \leq 0$  or  $\operatorname{Im} \tau(-\xi + 2\xi^0) \leq 0$ . In view of the hypothesis that  $\operatorname{Im} \tau(\xi) \geq 0$  for all  $\xi \in \Omega$ , we must have  $\partial_{\xi_i} \operatorname{Im} \tau(\xi^0) = 0$  for each  $i = 1, \dots, n$ . In conclusion,  $\operatorname{Im} \tau(\xi) = O(|\xi - \xi^0|^2)$  for all  $\xi \in B_\varepsilon(\xi^0)$ , which means that the zero is of order  $s \geq 2$ , and a similar argument shows that  $s$  must be even; also, this means that there exist  $c_0, c_1 > 0$  so that the above inequality holds for  $\xi \in B_\varepsilon(\xi^0)$ , proving the claim.

Now, we need the following result, which will be useful in the sequel. Moreover, we will give its further extension in Proposition 7.3 to deal with the setting of Theorem 2.15.

**Proposition 6.17.** *Let  $\phi : U \rightarrow \mathbb{R}$ ,  $U \subset \mathbb{R}^n$  open, be a continuous function and suppose  $\xi^0 \in U$  is such that  $\phi(\xi^0) = 0$  and such that  $\phi(\xi) > 0$  in a punctured open neighbourhood of  $\xi^0$ ,  $V \setminus \{\xi^0\}$ . Furthermore, assume that, for some  $s > 0$ , there exists a constant  $c_0 > 0$  such that, for all  $\xi \in V$ ,*

$$\phi(\xi) \geq c_0 |\xi - \xi^0|^s.$$

*Then, for any function  $a(\xi)$  that is bounded and compactly supported in  $U$ , and for all  $t \geq 0$ ,  $f \in C_0^\infty(\mathbb{R}^n)$ , and  $r \in \mathbb{R}$ , we have*

$$\int_V e^{-\phi(\xi)t} |\xi - \xi^0|^r |a(\xi)| |\widehat{f}(\xi)| d\xi \leq C(1+t)^{-(n+r)/s} \|f\|_{L^1}, \quad (6.20)$$

and

$$\|e^{-\phi(\xi)t} |\xi - \xi^0|^r a(\xi) \widehat{f}(\xi)\|_{L^2(V)} \leq C(1+t)^{-r/s} \|f\|_{L^2}. \quad (6.21)$$

The constant  $C$  depends on  $U, V, c_0$  and  $\|a\|_{L^\infty}$ , but not on the position of  $\xi_0$ .

First, we establish a straightforward result that is useful in proving each of these estimates:

**Lemma 6.18.** *For each  $\rho, M \geq 0$  and  $\sigma, c > 0$  there exists  $C \equiv C_{\rho, \sigma, M, c} \geq 0$  such that, for all  $t \geq 0$ , we have*

$$\int_0^M x^\rho e^{-cx^\sigma t} dx \leq C(1+t)^{-(\rho+1)/\sigma} \text{ and } \sup_{0 \leq x \leq M} x^\rho e^{-cx^\sigma t} \leq C(1+t)^{-\rho/\sigma}.$$

*Proof.* For  $0 \leq t \leq 1$ , each is clearly bounded: the first by  $\frac{M^{\rho+1}}{\rho+1}$  and the second by  $M^\rho$ . For  $t > 1$ , set  $y = xt^{1/\sigma}$ ; with this substitution, the first becomes

$$\int_0^{Mt^{1/\sigma}} y^\rho t^{-\rho/\sigma} e^{-cy^\sigma} t^{-1/\sigma} dy \leq t^{-(\rho+1)/\sigma} \int_0^\infty y^\rho e^{-cy^\sigma} dy,$$

while the second becomes

$$\sup_{0 \leq y \leq Mt^{1/\sigma}} y^\rho t^{-\rho/\sigma} e^{-cy^\sigma} \leq t^{-\rho/\sigma} \sup_{y \geq 0} y^\rho e^{-cy^\sigma};$$

These estimates imply those of Lemma 6.18 since both the integral and the supremum in the right hand sides are bounded.  $\square$

*Proof of Proposition 6.17.* As for the proof of (6.20), since  $a(\xi)$  is bounded in  $U$  by assumption, we have

$$\int_V e^{-\phi(\xi)t} |\xi - \xi^0|^r |a(\xi)| |\widehat{f}(\xi)| d\xi \leq C \int_{V'} e^{-\phi(\xi)t} |\xi - \xi^0|^r |\widehat{f}(\xi)| d\xi,$$

where  $V' = V \cap \text{supp } a$ ; this, in turn, can be estimated in the following manner using the hypothesis on  $\phi(\xi)$  and Hölder's inequality:

$$\begin{aligned} \int_{V'} e^{-\phi(\xi)t} |\xi - \xi^0|^r |\widehat{f}(\xi)| d\xi &\leq C \int_{V'} e^{-c_0|\xi - \xi^0|^s t} |\xi - \xi^0|^r |\widehat{f}(\xi)| d\xi \\ &\leq C \int_{V'} e^{-c_0|\xi - \xi^0|^s t} |\xi - \xi^0|^r d\xi \|\widehat{f}\|_{L^\infty(V')} . \end{aligned}$$

Then, transforming to polar coordinates and using the Hausdorff–Young inequality, we find that, for some  $R > 0$  (chosen so that  $V' \subset B_R(\xi^0)$ , which is possible since  $a(\xi)$  is compactly supported), we have

$$\int_{V'} e^{-c_0|\xi - \xi^0|^s t} |\xi - \xi^0|^r d\xi \|\widehat{f}\|_{L^\infty(V')} \leq C \int_{\mathbb{S}^{n-1}} \int_0^R |\eta|^{r+n-1} e^{-c_0|\eta|^s t} d|\eta| d\omega \|f\|_{L^1(\mathbb{R}^n)} .$$

Finally, by the first part of Lemma 6.18, we find

$$\begin{aligned} \int_V e^{-\phi(\xi)t} |\xi - \xi^0|^r |a(\xi)| |\widehat{f}(\xi)| d\xi &\leq C \int_0^R y^{r+n-1} e^{-c_0 y^s t} dy \|f\|_{L^1(\mathbb{R}^n)} \\ &\leq C(1+t)^{-(n+r)/s} \|f\|_{L^1} . \end{aligned}$$

This completes the proof of the first part.

Now let us look at the second part. By the second part of Lemma 6.18, we get

$$\begin{aligned} \|e^{-\phi(\xi)t} |\xi - \xi^0|^r a(\xi) \widehat{f}(\xi)\|_{L^2(V)}^2 &\leq \int_{V'} e^{-2c_0|\xi - \xi^0|^s t} |\xi - \xi^0|^{2r} |\widehat{f}(\xi)|^2 d\xi \\ &\leq C(1+t)^{-2r/s} \int_{V'} e^{-c_0|\xi - \xi^0|^s t} |\widehat{f}(\xi)|^2 d\xi . \end{aligned}$$

Now, it follows that

$$\int_{V'} e^{-c_0|\xi - \xi^0|^s t} |\widehat{f}(\xi)|^2 d\xi \leq \sup_{V'} |e^{-c_0|\xi - \xi^0|^s t}| \|\widehat{f}\|_{L^2(V')}^2 \leq C \|f\|_{L^2}^2 .$$

Together these give the required estimate (6.21).  $\square$

So, using this proposition, we have, for all  $t \geq 0$ , and sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} \left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right\|_{L^\infty(\mathbb{R}_x^n)} \\ \leq \int_{B_\varepsilon(\xi^0)} e^{-\text{Im } \tau(\xi)t} |a(\xi)| |\tau(\xi)|^r |\xi|^\alpha |\widehat{f}(\xi)| d\xi \leq C(1+t)^{-n/s} \|f\|_{L^1} , \end{aligned}$$

and, using the Plancherel's theorem, we get

$$\begin{aligned} \left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} \\ = C \|e^{i\tau(\xi)t} \tau(\xi)^r \xi^\alpha a(\xi) \widehat{f}(\xi)\|_{L^2(B_\varepsilon(\xi^0))} \leq C \|f\|_{L^2} ; \end{aligned}$$

here we have used that  $|\xi|^{|\alpha|}|\tau(\xi)|^r \leq C$  on  $B_\epsilon(\xi^0)$  for  $r \in \mathbb{N}$ ,  $\alpha$  a multi-index.

Thus, by Theorem 6.4, for all  $t \geq 0$ , we get

$$\left\| D_t^r D_x^\alpha \int_{B_\epsilon(\xi^0)} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right\|_{L^p(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^q}, \quad (6.22)$$

where  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . This completes the proof of Theorem 2.16 for roots meeting the axis with finite order and no multiplicities.

**Remark 6.19.** If  $\xi^0 = 0$ , then Proposition 6.17 further tells us that

$$\left\| D_t^r D_x^\alpha \int_{B_\epsilon(0)} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q}) - \frac{|\alpha|}{s}} \|f\|_{L^p}.$$

If, in addition, we have  $|\tau(\xi)| \leq c_1 |\xi - \xi^0|^{s_1}$ , for  $\xi$  near  $\xi^0$ , then we also get

$$\left\| D_t^r D_x^\alpha \int_{B_\epsilon(\xi^0)} e^{i(x \cdot \xi + \tau(\xi)t)} a(\xi) \widehat{f}(\xi) d\xi \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q}) - \frac{rs_1}{s}} \|f\|_{L^p}.$$

If both assumptions hold, we get the improvement from both cases, which is the estimate by  $C(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q}) - \frac{|\alpha|}{s} - \frac{rs_1}{s}}$ .

From this, we obtain the statement of Theorem 2.16 in the frequency region  $B_\epsilon(\xi^0)$ . Since there are only finitely many such points by hypothesis (H2) of Theorem 2.16, hypothesis (H1) guarantees that on the complement of their neighborhoods we have  $\text{Im } \tau_k > 0$ . There we can apply Theorems 2.1 and 2.2 to get the exponential decay. It may happen that the roots are multiple, but Theorem 2.2 provides the required estimate in such cases as well. The Sobolev orders in Theorem 2.16 come from large frequencies as given in Theorem 2.1. This completes the proof of Theorem 2.16 and of Remark 2.17.

## 6.12 Phase function lies on the real axis

As in the case of large  $|\xi|$ , we can subdivide into several subcases:

- (i)  $\det \text{Hess } \tau(\xi) \neq 0$ ;
- (ii)  $\det \text{Hess } \tau(\xi) = 0$  and  $\tau(\xi)$  satisfies the convexity condition;
- (iii) the general case when  $\det \text{Hess } \tau(\xi) = 0$ .

For the first case, the approach used in Section 6.5 can be used here also, since there we do not use that  $|\xi|$  is large other than to ensure that  $\tau(\xi)$  was smooth; here, we are away from multiplicities, so that still holds. Therefore, the conclusion is the same, giving Theorem 2.3.

The other two cases are considered in the next section alongside the case where there are multiplicities since it is important precisely how the integral is split up for such cases.

## 7 Estimates for bounded frequencies around multiplicities

Finally, let us turn to finding estimates for the first term of (6.7), which we may write in the form

$$\int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi,$$

where the characteristic roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  coincide in a set  $\mathcal{M} \subset \Omega$  of codimension  $\ell$  (in the sense of Section 2.1),  $\Omega \subset \mathbb{R}^n$  is a bounded open set and  $\chi \in C_0^\infty(\Omega)$ .

As before, we must consider the cases where the image of the phase function(s) either lie on the real axis, are separated from the real axis or meet the real axis. One additional thing to note in this case is that in principle the order of contact at points of multiplicity may be infinite as the roots are not necessarily analytic at such points; we have no examples of such a situation occurring, so it is not worth studying too deeply unless such an example can be found—for now, we can use the same technique as if the point(s) were points where the roots lie entirely on the real axis, and the results in these two situations are given together in Theorem 2.18. We study this very briefly nevertheless to ensure the completeness of the obtained results.

Unlike in the case away from multiplicities of characteristic roots, we have no explicit representation for the coefficients  $A_j^k(t, \xi)$  (as we have in Lemma 6.1 away from the multiplicities), which in turn means we cannot split this into  $L$  separate integrals. To overcome this, we first show, in Section 7.1, that a useful representation for the above integral does exist that allows us to use techniques from earlier. Using this alternative representation, it is a simple matter to find estimates in the case where the image of the set  $\mathcal{M}$  mapped by the characteristic roots is separated from the real axis (this is Theorem 2.2) and when it arises on the real axis as a result of all the roots meeting the axis with finite order, and these are done in Sections 7.2 and 7.3, respectively.

The situations where the roots meet on the real axis and at least one has a zero of infinite order there (either because it fully lies on the axis, or because it meets the axis with infinite order) is slightly more complicated; this is discussed in Section 7.4.

### 7.1 Resolution of multiple roots

In this section, we find estimates for

$$\sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi),$$

corresponding to (2.8), where  $\tau_1(\xi), \dots, \tau_L(\xi)$  coincide in a set  $\mathcal{M}$  of codimension  $\ell$ . For simplicity, first consider the simplest case of two roots intersecting at a single point, so that we have  $L = 2$  and  $\mathcal{M} = \{\xi^0\}$ ; the general case works in a similar way, and we shall show how it differs below. So, assume

$$\tau_1(\xi^0) = \tau_2(\xi^0) \text{ and } \tau_k(\xi^0) \neq \tau_1(\xi^0) \text{ for } k = 3, \dots, m;$$



by continuity, there exists a ball of radius  $\varepsilon > 0$  about  $\xi^0$ ,  $B_\varepsilon(\xi^0)$ , in which the only root which coincides with  $\tau_1(\xi)$  is  $\tau_2(\xi)$ . Then:

**Lemma 7.1.** *For all  $t \geq 0$  and  $\xi \in B_\varepsilon(\xi^0)$ , we have*

$$\left| \sum_{k=1}^2 e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right| \leq C(1+t)e^{-\min(\operatorname{Im} \tau_1(\xi), \operatorname{Im} \tau_2(\xi))t}, \quad (7.1)$$

where the minimum is taken over  $\xi \in B_\varepsilon(\xi^0)$ .

*Proof.* First, note that in the set

$$S := \{\xi \in \mathbb{R}^n : \tau_1(\xi) \neq \tau_k(\xi) \ \forall k = 2, \dots, m \text{ and } \tau_2(\xi) \neq \tau_l(\xi) \ \forall l = 3, \dots, m\}$$

the formula (6.4) is valid for  $A_j^1(\xi)$  and  $A_j^2(\xi)$ . Now, recall that the sum  $E_j(t, \xi) = \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi)$  is the solution to the Cauchy problem (6.2a), (6.2c), and thus is continuous; therefore, for all  $\eta \in \mathbb{R}^n$  such that  $\tau_1(\eta) \neq \tau_k(\eta)$  and  $\tau_2(\eta) \neq \tau_k(\eta)$  for  $k = 3, \dots, m$  (but allow  $\tau_1(\eta) = \tau_2(\eta)$ ), we have

$$\sum_{k=1}^2 e^{i\tau_k(\eta)t} A_j^k(t, \eta) = \lim_{\xi \rightarrow \eta} (e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)),$$

provided  $\xi$  varies in the set  $S$  (thus, ensuring  $e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)$  is well-defined). Hence, to obtain (7.1) for all  $\xi \in B_\varepsilon(\xi^0)$ , it suffices to show

$$|e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)| \leq C(1+t)e^{-\min(\operatorname{Im} \tau_1(\xi), \operatorname{Im} \tau_2(\xi))t}$$

for all  $t \geq 0$ ,  $\xi \in B'_\varepsilon(\xi^0) = B_\varepsilon(\xi^0) \setminus \{\xi^0\}$ .

Now, note the following trivial equality:

$$\begin{aligned} K_1 e^{iy_1} + K_2 e^{iy_2} &= K_1 e^{iy_2} e^{i(y_1-y_2)} + K_2 e^{iy_1} e^{-i(y_1-y_2)} \\ &= \frac{e^{i(y_1-y_2)} - e^{-i(y_1-y_2)}}{2} K_1 e^{iy_2} + \frac{e^{i(y_1-y_2)} + e^{-i(y_1-y_2)}}{2} K_1 e^{iy_2} \\ &\quad + \frac{e^{-i(y_1-y_2)} - e^{i(y_1-y_2)}}{2} K_2 e^{iy_1} + \frac{e^{-i(y_1-y_2)} + e^{i(y_1-y_2)}}{2} K_2 e^{iy_1} \\ &= \sinh(y_1 - y_2)[K_1 e^{iy_2} - K_2 e^{iy_1}] + \cosh(y_1 - y_2)[K_1 e^{iy_2} + K_2 e^{iy_1}]. \end{aligned}$$

Using this, we have, for all  $\xi \in B'_\varepsilon(\xi^0)$ ,  $t \geq 0$ ,

$$\begin{aligned} e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi) &= \sinh[(\tau_1(\xi) - \tau_2(\xi))t] (e^{i\tau_2(\xi)t} A_j^1(\xi) - e^{i\tau_1(\xi)t} A_j^2(\xi)) \\ &\quad + \cosh[(\tau_1(\xi) - \tau_2(\xi))t] (e^{i\tau_2(\xi)t} A_j^1(\xi) + e^{i\tau_1(\xi)t} A_j^2(\xi)). \end{aligned} \quad (7.2)$$

We estimate each of these terms:

- (a) “sinh” term: The first term is simple to estimate: since

$$\frac{\sinh[(\tau_1(\xi) - \tau_2(\xi))t]}{(\tau_1(\xi) - \tau_2(\xi))} \rightarrow t \text{ as } (\tau_1(\xi) - \tau_2(\xi)) \rightarrow 0,$$

or, equivalently, as  $\xi \rightarrow \xi^0$  through  $S$ , and  $A_j^k(\xi)(\tau_1(\xi) - \tau_2(\xi))$  is continuous in  $B_\varepsilon(\xi^0)$  for  $k = 1, 2$ , it follows that, for all  $\xi \in B'_\varepsilon(\xi^0)$ ,  $t \geq 0$ , we have

$$\begin{aligned} & |\sinh[(\tau_1(\xi) - \tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} - A_j^2(\xi)e^{i\tau_1(\xi)t})| \\ & \leq Ct[|e^{i\tau_2(\xi)t}| + |e^{i\tau_1(\xi)t}|] \leq Cte^{-\min(\operatorname{Im} \tau_1(\xi), \operatorname{Im} \tau_2(\xi))t}. \end{aligned} \quad (7.3)$$

- (b) “cosh” term: Estimating the second term is slightly more complicated. First, recall the explicit representation (6.4) for the  $A_j^k(\xi)$  at points away from multiplicities of  $\tau_k(\xi)$

$$A_j^k(\xi) = \frac{(-1)^j \sum_{1 \leq s_1 < \dots < s_{m-j-1} \leq m} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\prod_{l=1, l \neq k}^m (\tau_l(\xi) - \tau_k(\xi))}.$$

So, we can write

$$\begin{aligned} & \cosh[(\tau_1(\xi) - \tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} + A_j^2(\xi)e^{i\tau_1(\xi)t}) \\ & = \frac{\cosh[(\tau_1(\xi) - \tau_2(\xi))t]}{\prod_{k=3}^m (\tau_k(\xi) - \tau_1(\xi))(\tau_k(\xi) - \tau_2(\xi))} \frac{e^{i\tau_2(\xi)t} F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t} F_{j+1}^{2,1}(\xi)}{\tau_1(\xi) - \tau_2(\xi)}, \end{aligned}$$

where

$$F_i^{\rho, \sigma}(\xi) := \left( \sum_{1 \leq s_1 < \dots < s_{m-i} \leq m} \prod_{q=1}^{m-i} \tau_{s_q}(\xi) \right) \prod_{k=1, k \neq \rho, \sigma}^m (\tau_k(\xi) - \tau_\sigma(\xi)).$$

Now,  $(\cosh[(\tau_1(\xi) - \tau_2(\xi))t]) / (\prod_{k=3}^m (\tau_k(\xi) - \tau_1(\xi))(\tau_k(\xi) - \tau_2(\xi)))$  is continuous in  $S$ , hence it is bounded there, and, thus, absolutely converges to a constant,  $C \geq 0$  say, as  $\xi \rightarrow \xi^0$  through  $S$ . This leaves the  $[e^{i\tau_2(\xi)t} F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t} F_{j+1}^{2,1}(\xi)] / (\tau_1(\xi) - \tau_2(\xi))$  term.

For this, write

$$F_i^{\rho, \sigma}(\xi) = \sum_{\kappa=0}^{m-1} Q_{\kappa, i}^{\rho, \sigma}(\xi) \tau_\sigma(\xi)^\kappa,$$

where the  $Q_{\kappa, i}^{\rho, \sigma}(\xi)$  are polynomials in the  $\tau_k(\xi)$  for  $k \neq \rho, \sigma$  (which depend on  $i$ ); also, note  $Q_{\kappa, i}^{\rho, \sigma}(\xi) = Q_{\kappa, i}^{\sigma, \rho}(\xi)$ . Then, we have

$$\begin{aligned} & \frac{e^{i\tau_2(\xi)t} F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t} F_{j+1}^{2,1}(\xi)}{\tau_1(\xi) - \tau_2(\xi)} \\ & = \frac{\sum_{\kappa=0}^{m-1} [Q_{\kappa, j+1}^{1,2}(\xi)(\tau_2(\xi)^\kappa e^{i\tau_2(\xi)t} - \tau_1(\xi)^\kappa e^{i\tau_1(\xi)t})]}{\tau_1(\xi) - \tau_2(\xi)}. \end{aligned} \quad (7.4)$$

Let us show that this is continuous in  $B_\varepsilon(\xi^0)$  and is bounded absolutely by  $Cte^{-\min\{\lambda_1, \lambda_2\}t}$ : for  $y_1 \neq y_2$ , and for all  $r, s \in \mathbb{N}$ ,  $t \geq 0$ , we have

$$\begin{aligned} \frac{y_2^s y_1^r e^{iy_2 t} - y_1^s y_2^r e^{iy_1 t}}{y_1 - y_2} &= \\ &= \frac{y_2^s y_1^r (e^{iy_2 t} - e^{iy_1 t})}{y_1 - y_2} + \frac{y_2^s e^{iy_1 t} (y_1^r - y_2^r)}{y_1 - y_2} + \frac{e^{iy_1 t} y_2^r (y_2^s - y_1^s)}{y_1 - y_2}. \end{aligned}$$

Furthermore, for all  $y_1, y_2 \in \mathbb{C}$ ,  $t \in [0, \infty)$ ,  $s \in \mathbb{N}$ ,

$$\left| \frac{e^{iy_2 t} - e^{iy_1 t}}{y_1 - y_2} \right| \leq C_0 t e^{-\min(\operatorname{Im} y_1, \operatorname{Im} y_2)t} \quad \text{and} \quad \left| \frac{y_1^s - y_2^s}{y_1 - y_2} \right| \leq C_s,$$

for some constants  $C_0, C_s$ . Using these with  $y_1 = \tau_1(\xi)$ ,  $y_2 = \tau_2(\xi)$ ,  $r = \kappa$ , and  $s$  chosen appropriately for  $Q_{\kappa, j+1}^{1,2}(\xi)$ , the continuity and upper bound follow immediately. Thus, for all  $\xi \in B'_\varepsilon(\xi^0)$ ,  $t \geq 0$ ,

$$\begin{aligned} |\cosh[(\tau_1(\xi) - \tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} + A_j^2(\xi)e^{i\tau_1(\xi)t})| \\ \leq C t e^{-\min(\operatorname{Im} \tau_1(\xi), \operatorname{Im} \tau_2(\xi))t}. \end{aligned} \quad (7.5)$$

Combining (7.2), (7.3) and (7.5) we have (7.1), which completes the proof of the lemma.  $\square$

Now we show that a similar result holds in the general case: suppose the characteristic roots  $\tau_1(\xi), \dots, \tau_L(\xi)$ ,  $2 \leq L \leq m$ , coincide in a set  $\mathcal{M}$ , and that  $\tau_1(\xi) \neq \tau_k(\xi)$  for all  $\xi \in \mathcal{M}$  when  $k = L+1, \dots, m$ . By continuity, we may take  $\varepsilon > 0$  so that the set  $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \varepsilon\}$  contains no points  $\eta$  at which  $\tau_1(\eta), \dots, \tau_L(\eta) = \tau_k(\eta)$  for  $k = L+1, \dots, m$ . With this notation, we have:

**Lemma 7.2.** *For all  $t \geq 0$  and  $\xi \in \mathcal{M}^\varepsilon$ , we have the estimate*

$$\left| \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right| \leq C(1+t)^{L-1} e^{-t \min_{k=1, \dots, L} \operatorname{Im} \tau_k(\xi)}, \quad (7.6)$$

where the minimum is taken over  $\xi \in \mathcal{M}^\varepsilon$ .

Note that this estimate does not depend on the codimension of  $\mathcal{M}$ .

*Proof.* First note that, just as in the previous proof, for all  $\eta \in \mathbb{R}^n$  such that  $\tau_1(\eta), \dots, \tau_L(\eta) \neq \tau_k(\eta)$  when  $k = L+1, \dots, m$  (but allowing any or all of  $\tau_1(\eta), \dots, \tau_L(\eta)$  to be equal),

$$\sum_{k=1}^L e^{i\tau_k(\eta)t} A_j^k(t, \eta) = \lim_{\xi \rightarrow \eta} (e^{i\tau_1(\xi)t} A_j^1(\xi) + \dots + e^{i\tau_L(\xi)t} A_j^L(\xi)),$$

provided  $\xi$  varies the set  $S := \bigcup_{l=1}^L S_l$ , where

$$S_l := \{\xi \in \mathbb{R}^n : \tau_l(\xi) \neq \tau_k(\xi) \forall k \neq l\},$$

to ensure that each term of the sum on the right-hand side is well-defined. Note that Lemma 6.2 ensures every point in  $\mathcal{M}$  is the limit of a sequence of points in  $S$  in the case of differential operators. Thus, we must simply show, for all  $t \geq 0$ ,  $\xi \in (\mathcal{M}^\varepsilon)' = \mathcal{M}^\varepsilon \setminus \mathcal{M}$ , that we have the estimate

$$|e^{i\tau_1(\xi)t} A_j^1(\xi) + \dots + e^{i\tau_L(\xi)t} A_j^L(\xi)| \leq C(1+t)^{L-1} e^{-t \min_{k=1,\dots,L} \operatorname{Im} \tau_k(\xi)}.$$

Now, we claim that we can write  $\sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi)$ , for  $\xi \in (\mathcal{M}^\varepsilon)'$  and  $t \geq 0$ , as a sum of terms involving products of  $\frac{(L-1)L}{2}$  sinh and cosh terms of differences of coinciding roots; to clarify, (7.2) is this kind of representation for  $L = 2$ , while for  $L = 3$ , we want sums of terms such as

$$\sinh[\alpha_1(\tau_1(\xi) - \tau_2(\xi))t] \cosh[\alpha_2(\tau_1(\xi) - \tau_3(\xi))t] \sinh[\alpha_3(\tau_2(\xi) - \tau_3(\xi))t],$$

where the  $\alpha_i$  are appropriately chosen constants; incidentally, a comparison to the  $L = 2$  case suggests that the term above is multiplied by

$$(A_j^1(\xi)e^{i\tau_2(\xi)t} - A_j^2(\xi)e^{i\tau_1(\xi)t})$$

in the full representation.

To show this, we do induction on  $L$ ; Lemma 7.1 gives us the case  $L = 2$  (note that the proof holds with  $\xi^0$  and  $B_\varepsilon(\xi^0)$  replaced throughout by  $\mathcal{M}$  and  $\mathcal{M}^\varepsilon$ , respectively). Assume there is such a representation for  $L = K \leq m - 1$ . Observe,

$$\begin{aligned} \sum_{k=1}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) &= \frac{1}{K} \sum_{k=1}^K e^{i\tau_k(\xi)t} A_j^k(\xi) + \frac{1}{K} \sum_{k=1, k \neq K}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) \\ &\quad + \dots + \frac{1}{K} \sum_{k=2}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi); \end{aligned}$$

by the induction hypothesis, there is a representation for each of these terms by means of products of  $\frac{(K-1)K}{2}$

$$\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ and } \cosh[\beta_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ terms,}$$

where  $1 \leq k, l \leq K + 1$  and the  $\alpha_{k,l}, \beta_{k,l}$  are some non-zero constants. Next, note that we can write  $(\tau_1(\xi) - \tau_2(\xi))$  (or, indeed, the difference of any pair of roots from  $\tau_1(\xi), \dots, \tau_{K+1}(\xi)$ ) as a linear combination of the  $\frac{K(K+1)}{2}$  differences  $\tau_k(\xi) - \tau_l(\xi)$  such that  $1 \leq k < l \leq K + 1$ ; that is

$$\sinh[\alpha_{1,2}(\tau_1(\xi) - \tau_2(\xi))t] = \sinh \left[ \sum_{1 \leq k < l \leq K+1} \alpha'_{k,l}(\tau_k(\xi) - \tau_l(\xi))t \right],$$

for some non-zero constants  $\alpha'_{k,l}$ ; similarly, there is such a representation for  $\cosh[\beta_{1,2}(\tau_1(\xi) - \tau_2(\xi))t]$ . Lastly, repeated application of the double angle formulae

$$\begin{aligned} \sinh(a \pm b) &= \sinh a \cosh b \pm \cosh a \sinh b, \\ \cosh(a \pm b) &= \cosh a \cosh b \pm \sinh a \sinh b, \end{aligned}$$

yields products of  $\frac{K(K+1)}{2}$  terms, which completes the induction step.

Now, as in the previous proof, each of these terms must be estimated. The key fact to observe is that

$$A_j^k(\xi) \prod_{l=1, l \neq k}^L (\tau_l(\xi) - \tau_k(\xi))$$

is continuous in  $\mathcal{M}^\varepsilon$  for all  $k = 1, \dots, L$ . Then, using the same arguments as for each of the terms in the earlier proof, and observing that the exponent of  $t$  is determined by the products involving either

- (a)  $(\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi)t)]/(\tau_k(\xi) - \tau_l(\xi)))$  terms, or
- (b)  $(e^{i\tau_k(\xi)t} - e^{i\tau_l(\xi)t})/(\tau_k(\xi) - \tau_l(\xi))$  terms (see (7.4)),

the estimate (7.6) is immediately obtained.  $\square$

## 7.2 Phase separated from the real axis: Theorem 2.2

We now turn back to finding  $L^p - L^q$  estimates for

$$\int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi,$$

when  $\tau_1(\xi), \dots, \tau_L(\xi)$  coincide in a set  $\mathcal{M}$  of codimension  $\ell$ ; choose  $\varepsilon > 0$  so that these roots do not intersect with any of the roots  $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$  in  $\mathcal{M}^\varepsilon$ . The set  $\Omega$  is bounded, and we may take  $\chi \in C_0^\infty(\mathcal{M}^\varepsilon)$ .

In this section (under assumptions of Theorem 2.2), we assume that there exists  $\delta > 0$  such that  $\text{Im } \tau_k(\xi) \geq \delta$  for all  $\xi \in \mathcal{M}^\varepsilon$ —so,  $\min_k \text{Im } \tau_k(\xi) \geq \delta > 0$ . For this, we use the same approach as in Section 6.10, but using Lemma 7.2 to estimate the sum. Firstly, the  $L^1 - L^\infty$  estimate:

$$\begin{aligned} & \left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^\infty(\mathbb{R}_x^n)} \\ &= \left\| \int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \tau_k(\xi)^r \right) \xi^\alpha \chi(\xi) \widehat{f}(\xi) dx \right\|_{L^\infty(\mathbb{R}_x^n)} \\ &\leq \max_k \sup_{\Omega} |\tau_k(\xi)|^r \int_{\mathcal{M}^\varepsilon} \left| \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right| |\xi|^{|\alpha|} |\widehat{f}(\xi)| dx \\ &\leq C(1+t)^{L-1} e^{-\delta t} \|\widehat{f}\|_{L^\infty(\mathcal{M}^\varepsilon)} \leq C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^1}. \end{aligned}$$

Similarly, the  $L^2 - L^2$  estimate:

$$\begin{aligned}
& \left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^2(\mathbb{R}_x^n)} \\
&= \left\| \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \tau_k(\xi)^r \right) \xi^\alpha \chi(\xi) \widehat{f}(\xi) \right\|_{L^2(\Omega)} \\
&\leq C(1+t)^{L-1} e^{-\delta t} \|\widehat{f}\|_{L^2(\Omega)} \leq C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^2}.
\end{aligned}$$

Then, Theorem 6.4 yields

$$\begin{aligned}
& \left\| D_t^r D_x^\alpha \left( \int_{\Omega} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) dx \right) \right\|_{L^q(\mathbb{R}_x^n)} \\
&\leq C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^p},
\end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ . Once again, we have exponential decay. This, together with (6.19) gives the statement when there are multiplicities away from the axis and completes the proof of Theorem 2.2.

### 7.3 Phase meeting the real axis: Theorem 2.15

We next look at the case where the characteristic roots  $\tau_1(\xi), \dots, \tau_L(\xi)$  that coincide in the  $C^1$  set  $\mathcal{M}$  of codimension  $\ell$  meet the real axis in  $\mathcal{M}$  with finite orders. If there are more points in  $\mathcal{M}$  at which the above roots meet the axis with finite order (or even with infinite order/lying on the axis), they may be considered separately in the same way (or using the method below where necessary), while away from such points, the roots are separated from the axis, and the previous arguments and results of Section 2.1 may be used.

Since the characteristic roots are not necessarily analytic (or even differentiable) in  $\mathcal{M}$ , we must look at each branch of the roots as they approach the real axis; set  $s_k$  to be the maximal order of the contact with the real axis for  $\tau_k(\xi)$ , that is, the maximal value for which there exist constant  $c_0 > 0$  such that

$$c_0 \operatorname{dist}(\xi, Z_k)^{s_k} \leq \operatorname{Im} \tau_k(\xi),$$

for all  $\xi$  sufficiently near  $Z_k$ , where  $Z_k = \{\xi \in \mathbb{R}^n : \operatorname{Im} \tau_k(\xi) = 0\}$ . By assumptions of Theorem 2.15, we have the estimate

$$c_0 \operatorname{dist}(\xi, \mathcal{M})^s \leq \operatorname{Im} \tau_k(\xi),$$

for some  $c_0 > 0$  and  $s \geq \max(s_1, \dots, s_L)$ , for  $\xi$  close to  $\mathcal{M}$ . We will need the following extension of Proposition 6.17. Its proof is similar to the proof of Proposition 6.17 if we consider the  $C^1$  coordinate system associated to  $\mathcal{M}$ . As usual  $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \epsilon\}$ .

**Proposition 7.3.** *Let  $U \subset \mathbb{R}^n$  be open and let  $\phi : U \rightarrow \mathbb{R}$  be a continuous function. Suppose  $\mathcal{M} \subset U$  is a  $C^1$  set of codimension  $\ell$  such that*

$$c_0 \operatorname{dist}(\xi, \mathcal{M})^s \leq \phi(\xi),$$

*for some  $c_0 > 0$ , and all  $\xi \in \mathcal{M}^\epsilon$  for sufficiently small  $\epsilon > 0$ . Then, for any function  $a(\xi)$  that is bounded and compactly supported in  $U$ , and for all  $t \geq 0$ ,  $f \in C_0^\infty(\mathbb{R}^n)$ , and  $r \in \mathbb{R}$ , we have*

$$\int_{\mathcal{M}^\epsilon} e^{-\phi(\xi)t} \operatorname{dist}(\xi, \mathcal{M})^r |a(\xi)| |\widehat{f}(\xi)| d\xi \leq C(1+t)^{-(\ell+r)/s} \|f\|_{L^1},$$

and

$$\|e^{-\phi(\xi)t} \operatorname{dist}(\xi, \mathcal{M})^r a(\xi) \widehat{f}(\xi)\|_{L^2(\mathcal{M}^\epsilon)} \leq C(1+t)^{-r/s} \|f\|_{L^2}.$$

The proof of this proposition is similar to the proof of Proposition 6.17 and is omitted. Theorem 2.15 states that we must have the estimate (2.12), which is

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\epsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^q(\mathbb{R}_x^n)} \\ \leq C(1+t)^{-\frac{\ell}{s} \left( \frac{1}{p} - \frac{1}{q} \right) + L-1} \|f\|_{L^p}. \end{aligned}$$

By Lemma 7.2 and Proposition 7.3, to estimate the sum in the amplitude, for all  $t \geq 0$ , we have

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\epsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^\infty(\mathbb{R}_x^n)} \\ \leq C \left\| \int_{\mathcal{M}^\epsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \tau_k(\xi)^r \right) \xi^\alpha \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^\infty(\mathbb{R}_x^n)} \\ \leq C \int_{\mathcal{M}^\epsilon} (1+t)^{L-1} e^{-t \min_{k=1, \dots, L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)| |\widehat{f}(\xi)| d\xi \\ \leq C(1+t)^{L-1-(\ell/s)} \|f\|_{L^1}. \end{aligned}$$

Also, using the Plancherel's theorem, we have

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\epsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^2(\mathbb{R}_x^n)} \\ = \left\| \int_{\mathcal{M}^\epsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \tau_k(\xi)^r \right) \xi^\alpha \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} \\ = \left\| \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \tau_k(\xi)^r \right) \xi^\alpha \chi(\xi) \widehat{f}(\xi) \right\|_{L^2(\mathcal{M}^\epsilon)} \\ \leq C(1+t)^{L-1} \left\| e^{-t \min_{k=1, \dots, L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)| |\widehat{f}(\xi)| \right\|_{L^2(\mathcal{M}^\epsilon)} \\ \leq C(1+t)^{L-1} \|f\|_{L^2}. \end{aligned}$$

Therefore, interpolation Theorem 6.4 yields, for all  $t \geq 0$ ,

$$\begin{aligned} \left\| D_t^r D_x^\alpha \left( \int_{\mathcal{M}^\varepsilon} e^{ix \cdot \xi} \left( \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi \right) \right\|_{L^q(\mathbb{R}_x^n)} \\ \leq C(1+t)^{-\frac{\ell}{s} \left( \frac{1}{p} - \frac{1}{q} \right) + L-1} \|f\|_{L^p}, \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq 2$ ; this, together with (6.22) proves Theorem 2.15 for roots meeting the axis with finite order.

## 7.4 Phase function on the real axis for bounded frequencies

Recall that in the division of the integral in Section 6.2, we have

$$\int_{B_{2M}(0)} e^{ix \cdot \xi} \left( \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \widehat{f}(\xi) d\xi,$$

which we then subdivide around and away from multiplicities. The cases where the root or roots are either separated from the real axis or meet it with finite order have already been discussed; here we shall complete the analysis by proving estimates for the situation where a root or roots lie on the real axis. These results can be also applied to the case of multiple roots.

We note that in the case of nonhomogeneous symbols this analysis is essential since time genuinely interacts with frequencies. Unlike in the case of homogeneous symbols in Section 1.2, where one could eliminate time completely from estimates by rescaling, here it is present in phases and amplitude and causes them to blow up even for low frequencies. Thus, we must carry out a detailed investigation of the structure of solutions for low frequencies, and it will be done in this section.

A number of estimates can be already obtained using our results on multiple roots from Section 7.1. To have any possibility of obtaining better estimates, we must impose additional conditions on the characteristic roots at low frequencies—for large  $|\xi|$ , these properties were obtained by using perturbation results, but naturally such results are no longer valid for  $|\xi| \leq M$ . Also, we can impose the convexity condition on the roots to obtain a better result than the general case. We will give different formulation of possible results in this section.

Again, throughout we assume that either  $\tau(\xi) \geq 0$  for all  $\xi$  or  $\tau(\xi) \leq 0$  for all  $\xi$ . The key point is to use a carefully chosen cut-off function to isolate the multiplicities and then use Theorem 4.8 or Theorem 5.3 to estimate the integrals where there are no multiplicities (and hence the coefficients  $A_j^k(t, \xi)$  are independent of  $t$ ) and use suitable adjustments around the singularities. For these purposes, let us first assume that the only multiplicity is at a point  $\xi^0 \in B_{2M}(0)$  and  $\tau_1(\xi^0) = \tau_2(\xi^0)$  are the only coinciding roots, and let  $\chi$  be a cut-off function around  $\xi^0$ . Then, we must consider the sum of the first two roots, where we have a multiplicity at  $\xi^0$ ,

$$I = \int_{B_{2M}(0)} e^{ix \cdot \xi} \left( \sum_{k=1}^2 e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right) \chi(\xi) \widehat{f}(\xi) d\xi, \quad (7.7)$$



and terms involving the remaining roots, which are all distinct,

$$II = \sum_{k=3}^m \int_{B_{2M}(0)} e^{i(x \cdot \xi + \tau_k(\xi))t} A_j^k(t, \xi) \chi(\xi) \widehat{f}(\xi) d\xi.$$

#### 7.4.1 Case of no multiplicities: Theorem 2.8

For the second of these integrals  $II$ , we wish to apply Theorem 4.8 if  $\tau_k(\xi)$  satisfies the convexity condition, and Theorem 5.3 otherwise.

In order to ensure the hypotheses of these theorems are satisfied, however, we need to impose an additional regularity condition on the behaviour of the characteristic roots for the relevant frequencies (i.e.  $\xi \in B_{2M}(0)$ ) to avoid pathological situations:

$$\text{Assume } |\partial_\omega \tau_k(\lambda\omega)| \geq C_0 \text{ for all } \omega \in \mathbb{S}^{n-1}, 2M \geq \lambda > 0. \quad (7.8)$$

Since this is satisfied for large  $|\xi|$  (see Proposition 3.8) and always satisfied for roots of operators with homogeneous symbols, it is quite a natural extra assumption.

The other hypotheses of these theorems hold: hypothesis (i) is satisfied because  $|\partial_\xi^\alpha \tau_k(\xi)| \leq C_\alpha$  for all  $\xi$  since the characteristic roots are smooth in  $\mathbb{R}^n$ ; hypothesis (ii) only requires information about high frequencies; and hypotheses (iv) holds by the same argument as for large  $|\xi|$ , where only Part II of Proposition 3.5 is needed, and that holds for all  $\xi \in \mathbb{R}^n$ . Also, the coefficients  $A_j^k(\xi)$  are smooth away from multiplicities, so the symbolic behaviour (i.e. decay, or bounded for small frequencies) holds.

Now  $L^1 - L^\infty$  and  $L^2 - L^2$  estimates can be found as in the case for large  $|\xi|$ , and the interpolation theorem used to give the desired results. Thus, with condition (7.8), we have proved the on axis, no multiplicities case of Theorem 2.8.

#### 7.4.2 Multiplicities: shrinking neighborhoods

Now we can turn to the other integral given by (7.7). Here we will analyse what happens in certain shrinking neighborhoods of multiplicities. First we will assume that only two roots intersect at an isolated point, and then we will indicate what happens in the general situation.

To continue the analysis of an isolated point of multiplicity as in (7.7), we introduce a cut-off function  $\psi \in C_0^\infty([0, \infty))$ ,  $0 \leq \psi(s) \leq 1$ , which is identically 0 for  $s > 1$  and 1 for  $s < \frac{3}{4}$ ; then (7.7) can be rewritten as the sum of two integrals  $I = I_1 + I_2$ , where

$$I_1 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(t|\xi - \xi^0|) \chi(\xi) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi,$$

$$I_2 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi.$$

We study  $L^1 - L^\infty$  estimates for  $I_1$  and  $L^2 - L^2$  estimates for both  $I_1$  and  $I_2$  in this section.

**$L^1 - L^\infty$  estimates:** For this, we use the resolution of multiplicities technique of Section 7.1. By Lemma 7.1, we have, in particular,

$$\left| \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \right| \leq C(1+t),$$

for  $|\xi - \xi^0| < t^{-1}$ . Now, we may estimate the integral using the compactness of the support of  $\psi(s)$ : for  $0 \leq t \leq 1$ ,  $I_1$  is clearly bounded; for  $t > 1$ , we have

$$\begin{aligned} |I_1| &\leq Ct \int_{\mathbb{R}^n} |\psi(t|\xi - \xi^0|)| |\widehat{f}(\xi)| d\xi \\ &= Ct^{1-n} \|\widehat{f}\|_{L^\infty} \int_{\mathbb{R}^n} \psi(|\eta|) d\eta \leq C(1+t)^{1-n} \|f\|_{L^1}. \end{aligned}$$

This argument can be extended to the case when  $L$  roots meet on a set of codimension  $\ell$ . In the following proposition we will change the notation for the cut-off function to avoid any confusion with point multiplicities in the case above.

**Proposition 7.4.** *Suppose that  $L$  roots intersect in a set  $\mathcal{M}$  of codimension  $\ell$ . Let  $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}) < \epsilon\}$ , and let  $\theta \in C_0^\infty(\mathcal{M}^\epsilon)$  for sufficiently small  $\epsilon > 0$ . Then we have the estimate*

$$\left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} \theta(t \text{dist}(\xi, \mathcal{M})) \sum_{k=1}^L A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi \right| \leq C(1+t)^{L-1-\ell}. \quad (7.9)$$

*Proof.* By using Lemma 7.2 in the (bounded) neighborhood  $\mathcal{M}^\epsilon$  of  $\mathcal{M}$ , we obtain

$$\left| \sum_{k=1}^L e^{i\tau_k(\xi)t} A_j^k(t, \xi) \right| \leq C(1+t)^{L-1}.$$

The size of the support of  $\theta(t \text{dist}(\xi, \mathcal{M}))$  can be bounded by  $(1+t)^{-\ell}$ , which implies estimate (7.9).  $\square$

**$L^2 - L^2$  estimates:** Let us now analyse the  $L^2$ -estimate. This analysis will apply not only in a shrinking, but in a fixed neighborhood of the set of multiplicities. We will discuss first the case of two roots intersecting at a point in more detail, thus analysing mainly integral  $I$  in (7.7). We can have several versions of  $L^2$ -estimates dependent on conditions on multiplicities and on the Cauchy data that we can impose. For example, by Lemma 7.1 and Plancherel's theorem we get

$$\|I\|_{L^2} \leq C(1+t) \|f\|_{L^2}. \quad (7.10)$$

On the other hand we can improve the time behaviour of the  $L^2$ -estimate (7.10) if we make additional regularity assumptions for the data. For example, we can

eliminate time from estimate (7.10) if we work in suitable Sobolev type spaces taking the singularity into account. Let us rewrite

$$\begin{aligned} I &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \chi(\xi) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \chi(\xi) \left[ (\tau_1(\xi) - \tau_2(\xi)) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \right] (\tau_1(\xi) - \tau_2(\xi))^{-1} \widehat{f}(\xi) d\xi. \end{aligned}$$

Using the representation from Lemma 6.1 we see that the expression in the square brackets is bounded. Hence by the Plancherel's theorem we get that

$$\|I\|_{L^2} \leq \|(\tau_1(\xi) - \tau_2(\xi))^{-1} \chi(\xi) \widehat{f}(\xi)\|_{L^2} = \|(\tau_1(D) - \tau_2(D))^{-1} \chi(D) f\|_{L^2}. \quad (7.11)$$

An example of this is the appearance of homogeneous Sobolev spaces for small frequencies in the analysis of the wave equations, or more general equations with homogeneous symbols. For example, in the case of the wave equation we have  $\tau_1(\xi) = |\xi|$  and  $\tau_2(\xi) = -|\xi|$ , so that (7.11) means that we have the low frequency estimate for the solution of the form

$$\|I\|_{L^2} \leq \|f\|_{\dot{H}^{-1}},$$

with the homogeneous Sobolev space  $\dot{H}^{-1}$ .

In the case of several roots intersecting in a set  $\mathcal{M}$ , we have similarly:

**Proposition 7.5.** *Suppose that  $L$  roots intersect in a set  $\mathcal{M}$ . Let  $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \text{dist}(\xi, \mathcal{M}) < \epsilon\}$ , and let  $\theta \in C_0^\infty(\mathcal{M}^\epsilon)$  for sufficiently small  $\epsilon > 0$ . Let  $J$  denote the part of solution corresponding to these roots microlocalised near the set  $\mathcal{M}$  of multiplicities:*

$$J(t, x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \theta(\xi) \sum_{k=1}^L A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi.$$

Then we have the estimate

$$\|J\|_{L^2(\mathbb{R}_x^n)} \leq C(1+t)^{L-1} \|f\|_{L^2(\mathbb{R}_x^n)}. \quad (7.12)$$

Moreover, let us assume without loss of generality that intersecting  $L$  roots are labeled by  $\tau_1, \dots, \tau_L$ . Then we also have

$$\left\| \prod_{1 \leq l < k \leq L} (\tau_l(D) - \tau_k(D))^{-1} J \right\|_{L^2(\mathbb{R}_x^n)} \leq C \|f\|_{L^2(\mathbb{R}_x^n)}. \quad (7.13)$$

Estimate (7.12) follows from Lemma 7.2 and Plancherel's theorem. Estimate (7.13) follows from Plancherel's theorem and formula (6.4).

Interpolating between Propositions 7.9 and 7.12, we can obtain different versions of the dispersive estimate in a region shrinking around  $\mathcal{M}$ , depending on whether we use (7.12) or (7.13).

### 7.4.3 Multiplicities: fixed neighborhoods

Here, for simplicity, we will concentrate on the case of two roots  $\tau_1$  and  $\tau_2$  intersecting at an isolated point  $\xi^0$ . We will discuss both  $L^1 - L^\infty$  and  $L^2 - L^2$  estimates under additional assumptions on the roots  $\tau_1$  and  $\tau_2$ .

**$L^1 - L^\infty$  estimates:** For  $I_2$  we are away from the singularity, so we can use that

$$\sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} = A_j^1(\xi) e^{i\tau_1(\xi)t} + A_j^2(\xi) e^{i\tau_2(\xi)t}.$$

Now, we would like to apply Theorem 4.8 (for the case where the root satisfies the convexity condition) and Theorem 5.3 (for the general case), as in the case of simple roots; however, the proximity of the multiplicity brings the additional cut-off function,  $(1 - \psi)(t|\xi - \xi^0|)$ , into play, and this depends on  $t$ . Therefore, the aforementioned results cannot be used directly. However, a similar result does hold, provided we impose some additional conditions, producing analogues of Theorems 4.8 and 5.3 in this case.

**Proposition 7.6.** *Let  $\chi \in C_0^\infty(\mathbb{R}^n)$ . Suppose  $\tau_k(\xi)$ ,  $k = 1, 2$ , satisfy the following assumptions on  $\text{supp } \chi$ :*

- (i) *for each multi-index  $\alpha$  there exists a constant  $C_\alpha > 0$  such that, for some  $\delta > 0$ ,*

$$|\partial_\eta^\alpha[(\nabla_\xi \tau_k)(\xi^0 + s\eta)]| \leq C_\alpha(1 + |\eta|)^{-|\alpha|}, \text{ for small } s > 0 \text{ and } |\eta| > \delta;$$

- (ii) *there exists a constant  $C_0 > 0$  such that  $|\partial_\omega \tau_k(\xi^0 + \lambda\omega)| \geq C > 0$  for all  $\omega \in \mathbb{S}^{n-1}$  and  $\lambda > 0$ ; in particular, each of the level sets*

$$\lambda \Sigma'_\lambda \equiv \Sigma_\lambda = \{\eta \in \mathbb{R}^n : \tau_k(\xi^0 + \eta) = \lambda\}$$

*is non-degenerate;*

- (iii) *there exists a constant  $R_1 > 0$  such that, for all  $\lambda > 0$ ,*

$$\Sigma'_\lambda := \frac{1}{\lambda} \Sigma_\lambda(\tau_k) \subset B_{R_1}(0).$$

*Furthermore, assume that  $A_j^k(\xi)$  satisfies the following condition: for each multi-index  $\alpha$  there exists a constant  $C_\alpha > 0$  such that*

- (iv) *we have the estimate*

$$|\partial_\eta^\alpha[A_j^k(\xi^0 + s\eta)]| \leq C_\alpha s^{-j}(1 + |\eta|)^{-j-|\alpha|}, \text{ for small } s > 0 \text{ and } |\eta| > \delta.$$

Finally, assume that  $\psi \in C_0^\infty((-\delta, \delta))$  is such that  $\psi(\sigma) = 1$  for  $|\sigma| \leq \delta/2$ . Then, the following estimate holds for all  $x \in \mathbb{R}^n$ ,  $t \geq 0$ :

$$\left| \sum_{k=1}^2 \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) d\xi \right| \leq C(1+t)^{j-n}, \quad (7.14)$$

for  $j \geq n - \frac{n-1}{\gamma}$ , where  $\gamma := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau_k))$ , if  $\tau_k(\xi)$  satisfies the convexity condition; and for  $j \geq n - \frac{1}{\gamma_0}$ , where  $\gamma_0 := \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\tau_k))$ , if it does not.

**Remark 7.7.** Conditions (i), (ii) and (iv) appear and are satisfied naturally when roots  $\tau_k(\xi)$  are homogeneous functions of order one—for example, the wave equation, or for homogeneous equations.

**Remark 7.8.** Assumption (iv) is needed because  $A_j^k(\xi)$  has a singularity at  $\xi^0$ , so we must ensure we are away from that—this is the role of the cut-off function  $(1 - \psi)(|\eta|)$  in this proposition;

**Remark 7.9.** As usual, for example in the convex case, taking  $j = n - \frac{n-1}{\gamma}$ , we get the time decay estimate

$$| \text{Left hand side of (7.14)} | \leq C(1+t)^{-\frac{n-1}{\gamma}}.$$

*Proof.* As before, cut-off near the wave front: let  $\kappa \in C_0^\infty(\mathbb{R}^n)$  be a cut-off function supported in  $B(0, r)$ . Then, consider

$$I_1(t, x) := \sum_{k=1}^2 \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \kappa(t^{-1}x + \nabla \tau_k(\xi)) d\xi,$$

and

$$I_2(t, x) := \sum_{k=1}^2 \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) (1 - \kappa)(t^{-1}x + \nabla \tau_k(\xi)) d\xi.$$

**Away from the wave front set:** First, we estimate  $I_2(t, x)$ ; we claim that

$$|I_2(t, x)| \leq C_r(1+t)^{j-n} \text{ for all } t > 0, x \in \mathbb{R}^n. \quad (7.15)$$

In order to show this, we consider each term of the sum separately,

$$I_2^k(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) (1 - \kappa)\left(\frac{x}{t} + \nabla \tau_k(\xi)\right) d\xi,$$

and imitate the proof of Lemma 4.10 (in which the corresponding term was estimated in Theorem 4.8), but noting that in place of  $g_R(\xi) \in C_0^\infty(\mathbb{R}^n)$  we have  $(1 - \psi)(t|\xi - \xi^0|)$ , which depends also on  $t$ ; in particular, this means that care must be taken when

carrying out the integration by parts when derivatives fall on  $(1 - \psi)(t|\xi - \xi^0|)$ . To take this into account, use the change of variables  $\xi = \xi^0 + t^{-1}\eta$ :

$$I_2^k(t, x) = e^{ix \cdot \xi^0} \int_{\mathbb{R}^n} e^{i(t^{-1}x \cdot \eta + \tau_k(\xi^0 + t^{-1}\eta)t)} A_j^k(\xi^0 + t^{-1}\eta) (1 - \psi)(|\eta|) \chi(\xi^0 + t^{-1}\eta) (1 - \kappa)(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)) t^{-n} d\eta.$$

Integrating by parts, with respect to  $\eta$  gives

$$I_2^k(t, x) = e^{ix \cdot \xi^0} t^{-n} \int_{\mathbb{R}^n} e^{i(t^{-1}x \cdot \eta + \tau_k(\xi^0 + t^{-1}\eta)t)} P^*[A_j^k(\xi^0 + t^{-1}\eta) (1 - \psi)(|\eta|) \chi(\xi^0 + t^{-1}\eta) (1 - \kappa)(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta))] d\eta,$$

where  $P^*$  is the adjoint operator to  $P = \frac{t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)|^2} \cdot \nabla_\eta$ ; this integration by parts is valid as  $|t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)| \geq r > 0$ , in the support of  $(1 - \kappa)(t^{-1}x + \nabla \tau_k(\xi^0 + t^{-1}\eta))$ . For suitable functions  $f \equiv f(\eta; x, t)$ , and  $\xi = \xi^0 + t^{-1}\eta$ , we have

$$\begin{aligned} P^*f &= \nabla_\eta \cdot \left[ \frac{t^{-1}x + (\nabla_\xi \tau_k)(\xi)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi)|^2} f \right] \\ &= \frac{\nabla_\eta \cdot (\nabla_\xi \tau_k)(\xi)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi)|^2} f + \frac{t^{-1}x + (\nabla_\xi \tau_k)(\xi)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi)|^2} \cdot \nabla_\eta f \\ &\quad - \frac{2(t^{-1}x + (\nabla_\xi \tau_k)(\xi)) \cdot [\nabla_\eta[(\nabla_\xi \tau_k)(\xi)] \cdot (t^{-1}x + (\nabla_\xi \tau_k)(\xi))]}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi)|^4} f. \end{aligned}$$

Comparing this to (4.16), observe that the first and third terms have one power of  $t$  fewer in the denominator due to the transformation; this is critical in this case where we are approaching a singularity in  $A_j^k(\xi^0 + t^{-1}\eta)$  when  $t \rightarrow \infty$ . By hypothesis (i), for  $\eta$  in the support of the integrand of  $I_2^k(t, x)$ , we get

$$\frac{\nabla_\eta \cdot [(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)]}{|t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)|^2} \leq C_r(1 + |\eta|)^{-1};$$

thus, we have

$$|P^*f| \leq C_r[(1 + |\eta|)^{-1}|f| + |\nabla_\eta f|].$$

In Lemma 4.10, we carried out this integration by parts repeatedly in order to estimate the integral. Here, however, note that differentiating  $(1 - \psi)(|\eta|)$  once is sufficient: by definition of  $\psi(s)$ ,

$$\partial_{\eta_j}[(1 - \psi)(|\eta|)] = -\frac{\eta_j}{|\eta|}(\partial_s \psi)(|\eta|)$$

is supported in  $\frac{3}{4} \leq |\eta| \leq 1$ , so

$$|\partial_{\eta_j}[(1 - \psi)(|\eta|)]| \leq C \mathbf{1}_{\frac{3}{4} \leq |\eta| \leq 1}(\eta),$$

where  $\mathbf{1}_{1 \geq |\eta| \geq 3/4}(\eta)$  denotes the characteristic function of  $\{\eta \in \mathbb{R}^n : 1 \geq |\eta| \geq 3/4\}$ ; hence, by hypothesis (iv), for large  $t$  we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left| \frac{t^{-1}x + (\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta)}{i|t^{-1}x + (\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta)|^2} \right| |A_j^k(\xi^0 + t^{-1}\eta)| |\partial_{\eta_j}[(1 - \psi)(|\eta|)]| \\
& \quad |\chi(\xi^0 + t^{-1}\eta)| |(1 - \kappa)(t^{-1}x + \nabla \tau_k(\xi^0 + t^{-1}\eta))| t^{-n} d\eta \\
& \leq C_r \int_{\frac{3}{4} \leq |\eta| \leq 1} |A_j^k(\xi^0 + t^{-1}\eta)| t^{-n} d\eta \\
& \leq C_r t^j \int_{\frac{3}{4} \leq |\eta| \leq 1} \frac{1}{(1 + |\eta|)^j} t^{-n} d\eta \leq C_r t^{j-n}, \tag{7.16}
\end{aligned}$$

which is the desired estimate (7.15).

On the other hand, if, when integrating by parts, the derivative does not fall on  $\psi(|\eta|)$ , we use a similar argument to that in the earlier proof; let us look at the effect of differentiating each of the other terms: in the support of  $\psi(|\eta|)$ , for each multi-index  $\alpha$  and  $t > 0$ ,

- $|\partial_{\eta}^{\alpha}[A_j^k(\xi^0 + t^{-1}\eta)]| \leq C_{\alpha} t^j (1 + |\eta|)^{-j-|\alpha|}$  by hypothesis (iv);
- $|\partial_{\eta}^{\alpha}[\chi(\xi^0 + t^{-1}\eta)]| \leq C_{\alpha} (1 + |\eta|)^{-|\alpha|}$ : for  $\alpha = 0$ , take  $C_{\alpha} = 1$ ; for  $|\alpha| \geq 1$ , note that

$$\partial_{\eta}^{\alpha}[\chi(\xi^0 + t^{-1}\eta)] = t^{-|\alpha|}(\partial_{\xi}^{\alpha}\chi)(\xi^0 + t^{-1}\eta),$$

and that  $(\partial_{\xi}^{\alpha}\chi)(\xi^0 + t^{-1}\eta)$  is supported in  $N \leq |\xi^0 + t^{-1}\eta| \leq 2N$ , so  $t^{-1} \leq C_{N, \xi^0} |\eta|^{-1}$ ;

- $|\partial_{\eta}^{\alpha}[(1 - \kappa)(t^{-1}x + (\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta))]| \leq C_{\alpha} (1 + |\eta|)^{-|\alpha|}$ : obvious for  $\alpha = 0$ ; for  $|\alpha| \geq 1$ , note

$$\begin{aligned}
& \partial_{\eta}^{\alpha}[(1 - \kappa)(t^{-1}x + (\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta))] \\
& \quad = -(\partial_{\xi}^{\alpha} \kappa)(t^{-1}x + \nabla_{\xi} \tau_k(\xi)) \partial_{\eta}^{\alpha}[(\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta)],
\end{aligned}$$

which yields the desired estimate by hypothesis (i).

Summarising, this means

$$\begin{aligned}
& |(1 - \psi)(|\eta|) \partial_{\eta}^{\alpha}[A_j^k(\xi^0 + t^{-1}\eta) \chi(\xi^0 + t^{-1}\eta) (1 - \kappa)(t^{-1}x + (\nabla_{\xi} \tau_k)(\xi^0 + t^{-1}\eta))]| \\
& \leq C_r (1 + |\eta|)^{-j-|\alpha|} t^j \mathbf{1}_{|\eta| \geq 3/4}(\eta).
\end{aligned}$$

So, repeatedly integrating by parts we find that either a derivative falls on  $(1 - \psi)(|\eta|)$  (in which case a similar argument to that in (7.16) above works) or we eventually get the integrable function  $C t^j (1 + |\eta|)^{-n-1} \mathbf{1}_{|\eta| \geq 3/4}(\eta)$  as an upper bound; in either case, we have (7.15).

**On the wave front set:** Next, we look at the term supported around the wave front set,  $I_1(t, x)$ . As in the case away from the wave front, set  $\xi = \xi^0 + t^{-1}\eta$ : consider, for  $k = 1, 2$ ,

$$I_1^k(t, x) := e^{ix \cdot \xi^0} \int_{\mathbb{R}^n} e^{i(t^{-1}x \cdot \eta + \tau_k(\xi^0 + t^{-1}\eta)t)} A_j^k(\xi^0 + t^{-1}\eta) (1 - \psi)(|\eta|) \chi(\xi^0 + t^{-1}\eta) \kappa(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)) t^{-n} d\eta.$$

As in the proof of Theorems 4.8 and 5.3, let  $\{\Psi_\ell(\eta)\}_{\ell=1}^L$  be a conic partition of unity, where the support of  $\Psi_\ell(\eta)$  is a cone  $K_\ell$ , and each cone can be mapped by rotation onto  $K_1$ , which contains  $e_n = (0, \dots, 0, 1)$ . Then, it suffices to estimate

$$t^{-n} \int_{\mathbb{R}^n} e^{i(t^{-1}x \cdot \eta + \tau_k(\xi^0 + t^{-1}\eta)t)} A_j^k(\xi^0 + t^{-1}\eta) (1 - \psi)(|\eta|) \Psi_1(\eta) \chi(\xi^0 + t^{-1}\eta) \kappa(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)) d\eta,$$

for  $k = 1, 2$ .

Let us parameterise the cone  $K_1$ : it follows from hypothesis (ii) that each of the level sets

$$\Sigma_{\lambda, t} \equiv \{\eta \in \mathbb{R}^n : \tau_k(\xi^0 + t^{-1}\eta) = t^{-1}\lambda\}$$

is non-degenerate; so, for some  $U \subset \mathbb{R}^{n-1}$ , and smooth function  $h_k(t, \lambda, \cdot) : U \rightarrow \mathbb{R}$ ,

$$K_1 = \{(\lambda y, \lambda h_k(t, \lambda, y)) : \lambda > 0, y \in U\}.$$

If  $\tau_k(\xi)$  satisfies the convexity condition, then  $h_k$  is also a concave function in  $y$ . Now, we change variables  $\eta \mapsto (\lambda y, \lambda h_k(t, \lambda, y))$  and will often omit  $t$  from the notation of  $h_k$  since the dependence on  $t$  will be uniform. We obtain:

$$t^{-n} \int_0^\infty \int_U e^{i\lambda(t^{-1}x' \cdot y + t^{-1}x_n h_k(\lambda, y) + 1)} A_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) (1 - \psi)(\lambda|(y, h_k(\lambda, y))|) \Psi_1(\lambda(y, h_k(\lambda, y))) \chi(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \kappa(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))) \frac{d\eta}{d(\lambda, y)} d\lambda dy, \quad (7.17)$$

where we have used  $\tau_k(\xi^0 + t^{-1}(\lambda y, \lambda h_k(\lambda, y))) = t^{-1}\lambda$ . As in the earlier proofs, we ensure  $x_n$  is away from zero in the cone—this requires hypotheses (i) and (iii). So, in the general case, we can write this as, with  $\tilde{x} = t^{-1}x$ ,  $\tilde{\lambda} = \lambda \tilde{x}_n = \lambda t^{-1}x_n$ ,

$$t^{-n} \int_0^\infty \int_U e^{i\lambda x_n(t^{-1}x_n^{-1}x' \cdot y + t^{-1}h_k(\lambda, y) + \tilde{x}_n^{-1})} A_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) (1 - \psi)(\lambda|(y, h_k(\lambda, y))|) \Psi_1(\lambda(y, h_k(\lambda, y))) \chi(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \kappa(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))) \frac{d\eta}{d(\lambda, y)} d\lambda dy.$$



If the convexity condition holds, then, as in the proof of Theorem 4.8, we have the Gauss map

$$\underline{\mathbf{n}}_k : K_1 \cap \Sigma'_\lambda \rightarrow S^{n-1}, \quad \underline{\mathbf{n}}_k(\zeta) = \frac{\nabla_\zeta[\tau_k(\xi^0 + t^{-1}\zeta)]}{|\nabla_\zeta[\tau_k(\xi^0 + t^{-1}\zeta)]|} = \frac{(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\zeta)}{|(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\zeta)|},$$

and, as before, can define  $z_k(\lambda) \in U$  so that

$$\underline{\mathbf{n}}_k(z_k(\lambda), h_k(\lambda, z(\lambda))) = -x/|x|.$$

Then,

$$\frac{x'}{x_n} = -\nabla_y h_k(\lambda, z(\lambda)).$$

So, in this case, (7.17) becomes:

$$\begin{aligned} (I_1^k)' &:= t^{-n} \int_0^\infty \int_U e^{i\lambda x_n [-t^{-1}\nabla_y h_k(\lambda, z(\lambda)) \cdot y + t^{-1}h_k(\lambda, y) + \tilde{x}_n^{-1}]} \\ &\quad A_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1 - \psi)(\lambda|(y, h_k(\lambda, y))|) \Psi_1(\lambda(y, h_k(\lambda, y))) \\ &\quad \chi(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \kappa(\tilde{x} + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))) \frac{d\eta}{d(\lambda, y)} d\lambda dy, \end{aligned}$$

Let us estimate this integral in the case where the convexity condition holds. We have:

- The same argument as in the earlier proof (which uses hypothesis (ii)), shows

$$\left| \frac{d\eta}{d(\lambda, y)} \right| \leq C\lambda^{n-1}.$$

The constant  $C$  here is independent of  $t$ ;

- Now, with  $\tilde{A}_j^k(\nu) = A_j^k(\nu)\chi(\nu)\kappa(\tilde{x} + (\nabla_\xi \tau_k)(\nu))\Psi_1(\lambda(y, h_k(\lambda, y)))$ , where  $\nu = \xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))$ , we have

$$\begin{aligned} |(I_1^k)'| &\leq t^{j-n} \int_0^\infty \left| \int_U e^{i\lambda \tilde{x}_n [-(y-z(\lambda)) \cdot \nabla_y h_k(\lambda, z(\lambda)) + h_k(\lambda, y) + h_k(\lambda, z(\lambda))]} \right. \\ &\quad \left. t^{-j}\lambda^j \tilde{A}_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1 - \psi)(\lambda|(y, h_k(\lambda, y))|) dy \right| \lambda^{n-1-j} d\lambda. \end{aligned}$$

- Now, applying Theorem 4.1—this may be used due to the properties of  $A_j^k(\xi)$  and  $\tau_k(\xi)$  stated in hypotheses (iv) and (i)—we find that

$$\begin{aligned} &\left| \int_U e^{i\lambda \tilde{x}_n [-(y-z(\lambda)) \cdot \nabla_y h_k(\lambda, z(\lambda)) + h_k(\lambda, y) + h_k(\lambda, z(\lambda))]} \right. \\ &\quad \left. t^{-j}\lambda^j \tilde{A}_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1 - \psi)(\lambda|(y, h_k(\lambda, y))|) dy \right| \leq C\lambda^{j-n}\tilde{\chi}(\lambda), \end{aligned}$$

where  $\tilde{\chi}(\lambda)$  is a compactly supported smooth function that is zero in a neighbourhood of the origin.

- Hence,

$$|(I_1^k)'| \leq t^{j-n} \int_0^\infty \tilde{\chi}(\lambda) \lambda^{-1} d\lambda \leq C t^{j-n}.$$

Finally, the general case without convexity can be estimated in a similar way, with the necessary changes used in the proof of Theorem 5.3 to account for the change in the phase function—in particular, the use of the Van der Corput Lemma, Lemma 5.1, in place of Theorem 4.1. This completes the proof of (7.14).  $\square$

Using Proposition 7.6, it is clear that

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi \right\|_{L^\infty(\mathbb{R}_x^n)} \\ \leq C(1+t)^{-\frac{n-1}{\gamma}} \|f\|_{L^1} \end{aligned}$$

if the roots satisfy the convexity condition, and

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \sum_{k=1}^2 A_j^k(t, \xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi) d\xi \right\|_{L^\infty(\mathbb{R}_x^n)} \\ \leq C(1+t)^{-\frac{1}{\gamma_0}} \|f\|_{L^1} \end{aligned}$$

otherwise. In comparison to (6.16), here we have  $L^1$ -norms on the right hand sides, since  $\chi$  is a cut-off function to bounded frequencies.

Finally, we must consider the case where  $L$  roots intersect; the above proof can easily be adapted for such a case, giving corresponding results.

**$L^2 - L^2$  estimates:** For the  $L^2$ -estimates on the support of  $(1 - \psi)(t|\xi - \xi^0|) \chi(\xi)$  we only need assumption (iv) of Proposition 7.6 with  $\alpha = 0$  for the amplitude, namely that

$$|A_j^k(\xi^0 + s\eta)| \leq C_\alpha s^{-j} (1 + |\eta|)^{-j}, \text{ for small } s > 0 \text{ and } |\eta| > \delta. \quad (7.18)$$

Then, for the left hand side of (7.14), we have

$$\begin{aligned} & \left\| \sum_{k=1}^2 \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} \\ &= \left\| \sum_{k=1}^2 e^{i\tau_k(\xi)t} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \widehat{f}(\xi) \right\|_{L^2(\mathbb{R}_\xi^n)} \\ &\leq \|t^j (1 + |\eta|)^{-j} \widehat{f}(\xi^0 + t^{-1}\eta)\|_{L^2(\mathbb{R}_\eta^n)}, \end{aligned}$$

where we used Plancherel's theorem, (7.18), and the notation  $s = t^{-1}$ ,  $\xi = \xi^0 + t^{-1}\eta$ , so that  $\eta = t(\xi - \xi^0)$ . Then we can easily estimate

$$\begin{aligned}
\|t^j(1 + |\eta|)^{-j}\widehat{f}(\xi^0 + t^{-1}\eta)\|_{L^2(\mathbb{R}_\eta^n)} &= \|t^j(1 + t|\xi - \xi^0|)^{-j}\widehat{f}(\xi)\|_{L^2(\mathbb{R}_\xi^n)} \\
&= \|(t^{-1} + |\xi - \xi^0|)^{-j}\widehat{f}(\xi)\|_{L^2(\mathbb{R}_\xi^n)} \\
&\leq \| |\xi - \xi^0|^{-j}\widehat{f}(\xi)\|_{L^2(\mathbb{R}_\xi^n)} \\
&= \| |D - D_0|^{-j}f\|_{L^2(\mathbb{R}_x^n)},
\end{aligned}$$

where  $D - D_0$  is a Fourier multiplier with symbol  $\xi - \xi^0$ . So, we finally obtain the estimate

$$\begin{aligned}
\left\| \sum_{k=1}^2 \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} A_j^k(\xi) (1 - \psi)(t|\xi - \xi^0|) \chi(\xi) \widehat{f}(\xi) d\xi \right\|_{L^2(\mathbb{R}_x^n)} \\
\leq C \| |D - D_0|^{-j}f\|_{L^2(\mathbb{R}_x^n)}.
\end{aligned}$$

In the case of equations with homogeneous symbols (like for the wave equation), when roots are homogeneous, we have  $\xi^0 = 0$ , so that the right hand side becomes just the norm in the corresponding homogeneous Sobolev space.

Due to the earlier bound near the multiplicity, we can combine the results with the interpolation Theorem 6.4.

## 8 Examples and extensions

Theorem 2.18 gives estimates for operators provided the characteristic roots satisfy certain hypotheses. However, in order to test the validity of such an estimate for an arbitrary linear, constant coefficient  $m^{\text{th}}$  order strictly hyperbolic operator with lower order terms, it is desirable to find conditions on the structure of the lower order terms under which certain conditions for the characteristic roots hold. For the case  $m = 2$ , a complete characterisation can be given, and some extension of this is discussed in Section 8.1. However, for large  $m$ , it is difficult to do such an analysis, as no explicit formulae are available in general; nevertheless, certain conditions can be found that do make the task of checking the conditions of the characteristic roots, and these are discussed in Section 8.2, where a method is also given that can be used to find many examples. Finally, in Section 8.5, we give a few applications of these results.

### 8.1 Wave equation with mass and dissipation

As an example of how to use Theorem 2.18, here we will show that we can still have time decay of solutions if we allow the negative mass but exclude certain low frequencies for Cauchy data. This is given in (8.1) below. In the case of the negative mass and positive dissipation, there is an interplay between them with frequencies that we are going to exhibit. The usual non-negative and also time dependent mass and dissipation with oscillations have been considered before, even with oscillations. See, for example, [HR03] and references therein.

Let us consider second order equations of the following form

$$\begin{cases} \partial_t^2 u - c^2 \Delta u + \delta \partial_t u + \mu u = 0, \\ u(0, x) = 0, \quad u_t(0, x) = g(x). \end{cases}$$

Here  $\delta$  is the dissipation and  $\mu$  is the mass. For simplicity, the first Cauchy data is taken to be zero. The general case when both Cauchy data are non-zero can be treated in a similar way. Let us now apply Theorem 2.18 to the analysis of this equation. The associated characteristic polynomial is

$$\tau^2 - c^2 |\xi|^2 - i\delta\tau - \mu = 0,$$

and it has roots

$$\tau_{\pm}(\xi) = \frac{i\delta}{2} \pm \sqrt{c^2 |\xi|^2 + \mu - \delta^2/4}.$$

Now, we have the following well-known cases, which also correspond to different cases of Theorem 2.18:

- $\delta = \mu = 0$ . This is the wave equation.
- $\delta = 0, \mu > 0$ . This is the Klein–Gordon equation.
- $\mu = 0, \delta > 0$ . This is the dissipative wave equation.
- $\delta < 0$ . In this case,  $\text{Im } \tau_{-}(\xi) \leq \frac{\delta}{2} < 0$  for all  $\xi$ , hence we cannot expect any decay in general.
- $\delta > 0, \mu > 0$ . In this case the discriminant is always strictly greater than  $-\delta^2/4$ , and thus the roots always lie in the upper half plane and are separated from the real axis. So we have exponential decay.

Here is the main case for us, where we can show an interesting interplay between negative mass  $\mu < 0$  and how it is compensated by positive dissipation  $\delta > 0$  for different frequencies:

- dissipation  $\delta \geq 0$ , mass  $\mu < 0$ . In this case, note that  $\text{Im } \tau_{-}(\xi) \geq 0$  if and only if  $c^2 |\xi|^2 + \mu \geq 0$ , i.e.  $\text{Im } \tau_{-}(\xi) = 0$  for  $|\xi| = \sqrt{-\mu}/c$ . Therefore, the answer depends on the Cauchy data  $g$ . In particular, if  $\text{supp } \widehat{g}$  is contained in  $\{c^2 |\xi|^2 + \mu \geq 0\}$ , then we may get decay of some type. More precisely, let  $B(0, r)$  denote the open ball with radius  $r$  centred at the origin. Then we have:
  - if  $g$  is such that  $\text{supp } \widehat{g} \cap B(0, \frac{\sqrt{-\mu}}{c}) \neq \emptyset$ , then we have no decay;
  - if there is some  $\epsilon > 0$  such that  $\text{supp } \widehat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{-\mu}}{c} + \epsilon)$ , then the roots are either separated from the real axis (if  $\delta > 0$ ), and we get exponential decay, or lie on the real axis (if  $\delta = 0$ ), and we get Klein–Gordon type behaviour (since the Hessian of  $\tau$  is nonsingular).

- if, for all  $g$ ,  $\text{supp } \widehat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{-\mu}}{c}) = \left\{ |\xi| \geq \frac{\sqrt{-\mu}}{c} \right\}$ , then again we must consider  $\delta = 0$  and  $\delta > 0$  separately.

If  $\delta = 0$ , then the roots lie completely on the real axis, and they meet on the sphere  $|\xi| = \sqrt{-\mu}/c$ . It follows from (2.17) (which is justified in Proposition 7.4) with  $L = 2$  and  $\ell = 1$  that, although the representation of solution as a sum of Fourier integrals breaks down at the sphere, the solution is still bounded in a  $(1/t)$ -neighbourhood of the sphere. In its complement we can get the decay.

If  $\delta > 0$ , then the root  $\tau_-$  comes to the real axis at  $|\xi| = \frac{\sqrt{-\mu}}{c}$ , in which case we get the decay

$$\|u(t, \cdot)\|_{L^q} \leq C(1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \|g\|_{L^p}. \quad (8.1)$$

Indeed, in this case the order of the root  $\tau_-$  at the axis is one, i.e. estimate (2.16) holds with  $s = 1$ . Here  $1/p + 1/q = 1$  and  $1 \leq p \leq 2$ . Note also that compared to the case of no mass when  $\ell = n$ , now the codimension of the sphere  $\left\{ \xi \in \mathbb{R}^n : |\xi| = \frac{\sqrt{-\mu}}{c} \right\}$  is  $\ell = 1$ . We can apply the last case of Part II of Theorem 2.18 with  $L = 1$  and  $s = \ell = 1$  which gives estimate (8.1).

## 8.2 Higher order equations

Let us now derive a simple consequence of the stability condition of  $\text{Im } \tau_k(\xi) \geq 0$ , for all  $k = 1, \dots, m$  and  $\xi \in \mathbb{R}^n$ , for the coefficient of the  $D_t^{m-1}u$  term in (1.1). In fact, this coefficient plays an important role for higher order equations and can be compared with the dissipation term in the dissipative wave equation.

Let  $L = L(D_t, D_x)$  be an  $m^{\text{th}}$  order constant coefficient, linear strictly hyperbolic operator such that  $\text{Im } \tau_k(\xi) \geq 0$  for all  $k = 1, \dots, m$  and for all  $\xi \in \mathbb{R}^n$ . Recall that the characteristic polynomial corresponding to the principal part of  $L$  is of the form

$$L_m = L_m(\tau, \xi) = \tau^m + \sum_{k=1}^m P_k(\xi) \tau^{m-k} = 0,$$

where the  $P_k(\xi)$  are homogeneous polynomials of order  $k$ . Then, by the strict hyperbolicity of  $L$ ,  $L_m$  has real roots  $\varphi_1(\xi) \leq \varphi_2(\xi) \leq \dots \leq \varphi_m(\xi)$  (where the inequalities are strict when  $\xi \neq 0$ ). By the Viète formulae, observe that

$$P_1(\xi) = - \sum_{k=1}^m \varphi_k(\xi) \in \mathbb{R}. \quad (8.2)$$

On the other hand, the characteristic polynomial of the full operator is

$$L(\tau, \xi) = \tau^m + \sum_{k=1}^m P_k(\xi) \tau^{m-k} + \sum_{j=0}^{m-1} \sum_{|\alpha|+l=j} c_{\alpha,l} \xi^\alpha \tau^l = 0. \quad (8.3)$$

In particular, the coefficient of the  $\tau^{m-1}$  term is

$$P_1(\xi) + c_{0,m-1} = - \sum_{k=1}^m \tau_k(\xi), \quad (8.4)$$

where the  $\tau_k(\xi)$ ,  $k = 1, \dots, m$  are the roots of (8.3). Comparing (8.2) and (8.4), we see that  $\text{Im} \left( \sum_{k=1}^m \tau_k(\xi) \right) = -\text{Im } c_{0,m-1}$ . Therefore, since  $\text{Im } \tau_k(\xi) \geq 0$  for all  $k = 1, \dots, m$  and  $\xi \in \mathbb{R}^n$ , it follows that  $\text{Im } c_{0,m-1} \leq 0$ , or, equivalently,  $\text{Re } i c_{0,m-1} \geq 0$ . Furthermore, if  $\text{Im } c_{0,m-1} = 0$  then it must be the case that  $\text{Im } \tau_k(\xi) = 0$  for all  $\xi \in \mathbb{R}^n$  and  $k = 1, \dots, m$  since the characteristic roots are continuous. Hence we have shown the following:

**Proposition 8.1.** *Let  $L = L(D_t, D_x)$  be an  $m^{\text{th}}$  order linear constant coefficient strictly hyperbolic operator such that all the characteristic roots  $\tau_k(\xi)$ ,  $k = 1, \dots, m$ , satisfy  $\text{Im } \tau_k(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$ . Then the imaginary part of the coefficient of  $D_t^{m-1}u$  is non-positive. Furthermore, if in addition the (imaginary part of the) coefficient of  $D_t^{m-1}u$  is zero then each of the characteristic roots lie completely on the real axis.*

If we transform our operator back to the form  $L(\partial_t, \partial_x)$ , this result tells us that in order for the characteristic polynomial to be stable, that is for  $\text{Im } \tau_k(\xi) \geq 0$  for all  $k = 1, \dots, m$ ,  $\xi \in \mathbb{R}^n$ , it is necessary for the coefficient of  $\partial_t^{m-1}u$  to be non-negative; this is the case for the dissipative wave equation. In some sense this may be interpreted as a *higher order dissipation*, since it is necessary for the characteristic roots to behave geometrically like those of the wave equation with a dissipative term, where they lie in the half-plane  $\text{Im } z \geq 0$  and lie away from  $\text{Im } z = 0$  for large  $|\xi|$ .

In the next section, we look at the case where characteristic roots must lie completely on the real axis. First, though, let us consider one case where a root lies completely on the real axis but the coefficient  $c_{0,m-1}$  is nonzero,  $c_{0,m-1} \neq 0$ .

Consider a constant coefficient strictly hyperbolic operator of the form

$$L_m(\partial_t, \partial_x) + L_{m-1}(\partial_t, \partial_x) + L_{m-2}(\partial_t, \partial_x) = 0, \quad (8.5)$$

where  $L_r = L_r(\partial_t, \partial_x)$  denotes a homogeneous operator of degree  $r$  with real coefficients. This is an example of a hyperbolic triple, which will be discussed in more generality in Section 8.3. Furthermore, assume that  $L_{m-1}$  is not identically zero. Let  $\tau(\xi) \in \mathbb{R}$  be a characteristic root of (8.5) which lies completely on the real axis. So, denoting as usual  $D_{x_j} = -i\partial_{x_j}$ ,  $D_t = -i\partial_t$ , we have that  $\tau(\xi)$  is a root of

$$L_m(\tau, \xi) - iL_{m-1}(\tau, \xi) - L_{m-2}(\tau, \xi) = 0.$$

This means that  $L_{m-1}(\xi, \tau(\xi)) = 0$ , and so  $\tau(\xi)$  is homogeneous of order 1, and thus for such roots Theorem 2.18 applies to yield results similar to those described in Section 1.2.

### 8.3 Hyperbolic triples

We now turn to the case when all the characteristic roots lie completely on the real axis. This section is devoted to showing some more examples of appearances of real valued non-homogeneous roots and some sufficient conditions for this. In order to study this case we first recall some results of Volevich–Radkevich [VR03] on hyperbolic pairs and triples. Throughout this section only,  $L_r(\tau, \xi)$  denotes a homogeneous polynomial in  $\tau$  and  $\xi = (\xi_1, \dots, \xi_n)$  of order  $r$  such that  $L_r(\tau, i\xi)$  has real coefficients.

**Definition 8.2.** Suppose  $L_m = L_m(\tau, \xi)$  and  $L_{m-1} = L_{m-1}(\tau, \xi)$  are homogeneous polynomials as above. Furthermore, assume that the roots of  $L_m$ ,  $\tau_1(\xi), \dots, \tau_m(\xi)$ , and those of  $L_{m-1}$ ,  $\sigma_1(\xi), \dots, \sigma_{m-1}(\xi)$ , are real-valued (in which case we say  $L_m$  and  $L_{m-1}$  are hyperbolic polynomials). Then,  $(L_m, L_{m-1})$  is called a hyperbolic pair if (possibly after reordering)

$$\tau_1(\xi) \leq \sigma_1(\xi) \leq \tau_2(\xi) \leq \dots \leq \tau_{m-1}(\xi) \leq \sigma_{m-1}(\xi) \leq \tau_m(\xi). \quad (8.6)$$

If, in addition, the roots of  $L_m, L_{m-1}$  are pairwise distinct for  $\xi \neq 0$  (in which case they are called strictly hyperbolic polynomials) and the inequalities in (8.6) are all strict, then we say  $(L_m, L_{m-1})$  is a strictly hyperbolic pair.

**Definition 8.3.** Let

$$L_m = L_m(\tau, \xi), \quad L_{m-1} = L_{m-1}(\tau, \xi), \quad L_{m-2} = L_{m-2}(\tau, \xi)$$

be (homogeneous) hyperbolic polynomials. If  $(L_m, L_{m-1})$  and  $(L_{m-1}, L_{m-2})$  are both hyperbolic pairs then we say that  $(L_m, L_{m-1}, L_{m-2})$  is a hyperbolic triple. If, in addition, all the polynomials and all the pairs are strictly hyperbolic (in the sense of Definition 8.2) then  $(L_m, L_{m-1}, L_{m-2})$  is called a strictly hyperbolic triple.

**Theorem 8.4** ([VR03]). Suppose that  $(L_m, L_{m-1}, L_{m-2})$  is a strictly hyperbolic triple. Then  $L_m(\tau, \xi) + L_{m-1}(\tau, \xi) + L_{m-2}(\tau, \xi) \neq 0$  for all  $\text{Im } \tau \leq 0$ . Furthermore, any two of the polynomials  $L_m, L_{m-1}, L_{m-2}$  have no common purely imaginary zeros.

We also recall a theorem of Hermite (see, for example, [Nis00]):

**Theorem 8.5.** Suppose  $p_m(z)$ ,  $p_{m-1}(z)$  are real polynomials of degree  $m, m-1$ , respectively, and that all the zeros of  $p(z) = p_m(z) - ip_{m-1}(z)$  lie in the upper half-plane (that is, if  $p(z) = 0$  then  $\text{Im } z > 0$ ). Then all the zeros of  $p_m(z)$  and  $p_{m-1}(z)$  are real and distinct.

Now we will give some rather constructive examples of how non-homogeneous real roots may arise, and some sufficient conditions for this.

Assume that  $L$  is of the form  $L_m(D_t, D_x) + L_{m-2}(D_t, D_x)$ , where the  $L_r$  are as in Definition 8.3 and neither is identically zero. Suppose that there exists a homogeneous operator of order  $m-1$ ,  $L_{m-1}(D_t, D_x)$ , such that the characteristic polynomials  $L_m(\tau, \xi)$ ,  $L_{m-1}(\tau, \xi)$  and  $L_{m-2}(\tau, \xi)$  form a strictly hyperbolic triple. Then, by Theorem 8.4, we have

$$L_m(\tau, \xi) + L_{m-1}(\tau, \xi) + L_{m-2}(\tau, \xi) \neq 0 \text{ for } \text{Im } \tau \leq 0.$$

Thus, by Theorem 8.5, all the zeros of  $L_m(\tau, \xi) + L_{m-2}(\tau, \xi)$  are real, but clearly non-homogeneous if  $L_{m-2} \not\equiv 0$ . So, using this construction, we can obtain examples of operators for which all the characteristic roots lie completely on the imaginary axis (so that  $i\tau(\xi)$  are real, which would be the notation for the rest of this paper), but for which we cannot automatically expect the standard decay for homogeneous symbols to hold.

## 8.4 Strictly hyperbolic systems

Our results can also be used to establish  $L^p - L^q$  decay rates for strictly hyperbolic systems. Let us briefly sketch the reduction of systems to the situation covered by results of this paper. Let

$$iU_t = A(D)U, \quad U(0) = U_0,$$

be an  $m \times m$  first order strictly hyperbolic system of partial differential equations. That is, the associated system of polynomials may be written as  $A(\xi) = A_1(\xi) + A_0(\xi)$ , with  $A_1$  being positively homogeneous of order one in  $\xi$  and  $A_0(\xi) \in S_{1,0}^0(\mathbb{R}^n)$ . If  $A(\xi)$  is a matrix of first order polynomials, then  $A_0$  is constant. It is known that  $A(D)$  is hyperbolic if and only if  $\det A(D)$  is hyperbolic (see e.g. Atiyah, Bott and Gårding [ABG]). Moreover, if  $\det A_1(D)$  is strictly hyperbolic, then  $A(D)$  is strongly hyperbolic.

Now, the strict hyperbolicity of  $A(D)$  means that the roots  $\varphi_1(\xi), \dots, \varphi_m(\xi)$  of equation  $\det(\varphi I - A_1(\xi)) = 0$  are all real and distinct away from the origin. Denote the roots of the equation  $\det(\tau I - A(\xi)) = 0$  (which is an  $m^{\text{th}}$  order polynomial in  $\tau$  with smooth coefficients) by  $\tau_1(\xi), \dots, \tau_m(\xi)$ . Now, by analogy to the case of the  $m^{\text{th}}$  order scalar equation, we can, via perturbation methods, show that for large  $|\xi|$  the  $\tau_k(\xi)$  behave similarly to the  $\varphi_k(\xi)$ , in that they are distinct, analytic and belong to  $S_{1,0}^1(\mathbb{R}^n)$ . For bounded  $|\xi|$  we will need similar regularity assumptions on the characteristic roots  $\tau_k(\xi)$  as for the scalar equations. Furthermore, we assume that there exists  $Q \in S_{1,0}^0(\mathbb{R}^n)$  such that  $|\det Q(\xi)| \geq C > 0$  and such that

$$Q^{-1}AQ = \text{diag}(\tau_1(\xi), \dots, \tau_m(\xi)) =: T.$$

The existence of such  $Q$  is a very interesting question itself, especially in the presence of variable multiplicities, but we will not go into such details here. Now, we use the transformation  $U = Q(D)V$ , so that

$$U_t = QV_t \implies iQV_t = A(D)QV \implies iV_t = TV; \quad U(0) = QV(0).$$

This systems decouples into  $m$  independent scalar equations:

$$\partial_t V_k = \tau_k(D)V_k, \quad k = 1, \dots, m, \quad V_k(0) = (Q^{-1}U(0))_k$$

each of which is solved by

$$V_k(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau_k(\xi)t)} \widehat{V}_k(0, \xi) d\xi.$$



Now,  $Q \in S^0(\mathbb{R}^n)$ , so it is a bounded map  $L^q \rightarrow L^q$ ,  $1 < q < \infty$ , and we can get our estimates for  $V_k$  as in the case of  $m^{\text{th}}$  order scalar equations; thus, we can conclude that

$$\|U\|_{L^q} = \|QV\|_{L^q} \leq C\|V\|_{L^q} \leq CK(t)\|V\|_{L^p} = CK(t)\|Q^{-1}U\|_{L^p} \leq CK(t)\|U\|_{L^p},$$

where  $K(t)$  is as in Theorem 2.18.

## 8.5 Application to Fokker–Planck equation

The classical Boltzmann equation for the particle distribution function  $f = f(t, x, c)$ , where  $x, \mathbf{c} \in \mathbb{R}^n$ ,  $n = 1, 2, 3$ , is

$$(\partial_t + \mathbf{c} \cdot \nabla_x)f = S(f),$$

where  $S(f)$  is the so-called integral of collisions. The important special case of this equation is the Fokker–Planck equation for the distribution function of particles in Brownian motion, when the integral of collisions is linear and is given by

$$S(f) = \nabla_{\mathbf{c}} \cdot (\mathbf{c} + \nabla_{\mathbf{c}})f = \sum_{k=1}^n \partial_{c_k}(c_k + \partial_{c_k})f.$$

In this case the kinetic Fokker–Planck equations takes the form

$$\left( \partial_t + \sum_{k=1}^n c_k \partial_{x_k} \right) f(t, x, c) = \sum_{k=1}^n \partial_{c_k}(c_k + \partial_{c_k})f.$$

The Hermite–Grad method of dealing with Fokker–Planck equation consists in decomposing  $f(t, x, \cdot)$  in the Hermite basis, i.e. writing

$$f(t, x, c) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} m_\alpha(t, x) \psi^\alpha(c),$$

where  $\psi^\alpha(c) = (2\pi)^{-n/2} (-\partial_c)^\alpha \exp(-\frac{|c|^2}{2})$  are Hermite functions. They are derivatives of the Maxwell distribution  $\psi^0$  which annihilates the integral of collisions and form a complete orthonormal basis in the weighted Hilbert space  $L_w^2(\mathbb{R}^n)$  with weight  $w = 1/\psi^0$ . This decomposition <sup>2</sup> yields the infinite system

$$\partial_t m_\beta(t, x) + \beta_k \partial_{x_k} m_{\beta - e_k}(t, x) + \partial_{x_k} m_{\beta + e_k}(t, x) + |\beta| m_\beta(t, x) = 0.$$

The Galerkin approximation  $f^N$  of the solution  $f$  is

$$f^N(t, x, c) = \sum_{0 \leq |\alpha| \leq N} \frac{1}{\alpha!} m_\alpha(t, x) \psi^\alpha(c),$$

---

<sup>2</sup>Thus, the convergence of the series of such decomposition is understood as a convergence of the decomposition with respect to a basis in a Hilbert space.

with  $m(t, x) = \{m_\beta(t, x) : 0 \leq |\beta| \leq N\}$  being the unknown function of coefficients. For  $m(t, x)$  one obtains the following system of equations

$$D_t m(t, x) + \sum_j A_j D_{x_j} m(t, x) - iBm(t, x) = 0,$$

where  $B$  is a diagonal matrix,  $B_{\alpha, \beta} = |\alpha| \delta_{\alpha, \beta}$ , and the only non-zero elements of the matrix  $A_j$  are  $a_j^{\alpha - e_j, \alpha} = \alpha_j$ ,  $a_j^{\alpha + e_j, \alpha} = 1$ . Hence, the dispersion equation for the system is

$$P(\tau, \xi) \equiv \det(\tau I + \sum_j A_j \xi_j - iB) = 0, \quad (8.7)$$

which we will call the  $N^{th}$  Fokker–Planck polynomial, and we have, in particular,

$$P(\tau, 0) = \det(\tau I - iB) = \tau \prod_{j=1}^N (\tau - ji)^{\gamma_j}, \quad (8.8)$$

for some powers  $\gamma_j \geq 0$ . Properties of this polynomial  $P(\tau, \xi)$  have been extensively studied by Volevich and Radkevich in [VR04], who gave conditions and examples of situations when  $\text{Im } \tau_j(\xi) \geq 0$ , for all  $\xi \neq 0$ . They also described more general (necessary) conditions in terms of coefficients of  $P$ . See also [VR03, ZR04]. In our situation here we have to take additional care of possible multiple roots, as is done in Theorem 2.16.

From formula (8.8) it follows in particular that there is a single characteristic root at the origin. Let  $M = \prod_{j=1}^N j^{\gamma_j}$ .

Let us examine the structure of the operator  $P(\tau, \xi)$ . It is a polynomial of order  $m$  which can be written in the form

$$P(\tau, \xi) = \sum_{j=0}^m (-i)^{m-j} P_j(\tau, \xi),$$

with  $P_j$  being a homogeneous polynomial of order  $j$ . Moreover, we have

$$P_0 = 0, \quad P_1 = M\tau, \quad P_2 = M \sum_{k=2}^m \frac{1}{k-1} \tau^2 - M|\xi|^2.$$

The case  $n = 1$  was considered in [VR03], where one has  $M = N!$

Let  $P(\tau(\xi), \xi) = 0$ , where  $\tau(0) = 0$  is the simple root at the origin. Differentiation with respect to  $\tau$  yields  $\frac{\partial \tau}{\partial \xi}(0) = 0$ . Differentiating again we get

$$\frac{\partial^2 \tau}{\partial \xi^2}(0) = 2iI_n.$$

So, for small frequencies we obtain the decomposition

$$\text{Im } \tau(\xi) = 2|\xi|^2 + \dots + c(\log m) \|\xi\|^4 + \dots,$$

where

$$m = 1 + \gamma_1 + \dots + \gamma_N \approx c_n N,$$

and  $||\xi||^4$  denotes a fourth order polynomial in  $\xi$ . We also easily have a rough estimate for  $M$  of the form

$$N^N \preceq M \preceq (N!)^N, \quad (n \geq 2).$$

It follows then that for *small* frequencies we get the estimate

$$|m(t, x)| \leq C(1+t)^{-n/2} + Ce^{-\varepsilon(N)t},$$

where, in general, it may be that  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ . For *medium* frequencies we get exponential decay in view of the result of Theorem 2.1, also in the case when there are multiple characteristics, where we can use Theorem 2.2. Here, there is an additional polynomial growth with respect to time caused by the resolution procedure of Section 7.1, but this is compensated by the exponential decay given by characteristics with strictly positive imaginary part (see Theorem 2.2).

Let us discuss the situation with *large* frequencies. For operators of general form, away from points where roots coincide, the roots are analytic. For large  $|\xi|$ , perturbation arguments of Section 3 give properties of roots  $\tau_k(\xi)$  related to  $\varphi_k(\xi)$ , the characteristics of the principal part. Here  $\tau_k(\xi)$  and  $\phi_k(\xi)$  are defined as roots of equations  $P(\tau, \xi) = 0$  and its principal part  $P_m(\varphi, \xi) = 0$ , respectively. Let  $K$  be the maximal order of lower order terms. Then we can summarise the following properties of  $P$  established in Section 3:

- there are no multiple roots for large  $\xi$ ;
- $|\partial_\xi^\alpha \tau_k(\xi)| \leq C(1 + |\xi|)^{1-|\alpha|}$ , i.e.  $\tau_k \in S^1$ ;
- the exists  $\varphi_k$  such that  $|\partial^\alpha \tau_k(\xi) - \partial^\alpha \varphi_k(\xi)| \leq C(1 + |\xi|)^{K+1-m-|\alpha|}$ , for all  $\xi \in \mathbb{R}^n$  and all multi-indices  $\alpha$ ;
- Since  $\phi_k$  are real-valued, we get  $\text{Im } \tau_k \in S^{K+1-m}$ . In particular,  $\text{Im } \tau_k \in S^0$ .

The statements above are obtained by perturbation arguments and rely on the strict hyperbolicity of the principal part. However, this does not have to be the case for polynomials  $P$  that we obtain in the Galerkin approximation. Moreover, in general, it might happen that  $\text{Im } \tau_k(\xi) \rightarrow 0$  as  $|\xi| \rightarrow \infty$ , the case which is discussed in Section 6.8. To avoid these problems we impose the condition of strong stability. First, we will say that  $P(\tau, \xi)$  is a *stable* polynomial if its roots  $\tau(\xi)$  satisfy  $\text{Im } \tau(\xi) \geq 0$  for all  $\xi \in \mathbb{R}^n$ , and if  $\text{Im } \tau(\xi) = 0$  implies  $\xi = 0$ . Then we will say that  $P(\tau, \xi)$  is *strongly stable* if, moreover,  $\text{Im } \tau(\xi) = 0$  implies  $\xi = 0$  and  $\text{Re } \tau(\xi) = 0$ , and if its roots  $\tau(\xi)$  satisfy  $\liminf_{|\xi| \rightarrow \infty} \text{Im } \tau(\xi) > 0$ . Thus, the condition of strong stability means that the roots  $\tau(\xi)$  may become real only at the origin of the complex plane at  $\xi = 0$ , and that they do not approach the real axis asymptotically for large  $\xi$ .

In Section 8.3, as well as in [VR03, VR04], there are several sufficient conditions for the stability of hyperbolic polynomials. In this case we have a consequence of Theorem 2.16 and Remark 2.17 in the form of estimate (2.15):

**Corollary 8.6.** *Let  $P$  be a strongly stable polynomial with characteristic roots with non-negative imaginary parts. Let  $1 \leq p \leq 2$  and  $2 \leq q \leq \infty$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then the solution to Cauchy problem (2.1) satisfies dispersive estimate (2.15), i.e. we have*

$$\left\| D_t^r D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}_x^n)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right) - \frac{|\alpha|}{s} - \frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}},$$

with  $N_p \geq n(\frac{1}{p} - \frac{1}{q})$  for  $1 < p \leq 2$  and  $N_1 > n$  for  $p = 1$ .

From this, we can conclude the following estimates for solution to the Galerkin approximations of Fokker–Planck equation:

**Theorem 8.7.** *If the  $N^{\text{th}}$  Fokker–Planck polynomial  $P$  in (8.7) is strongly stable, we have the estimate*

$$\|f_N(t, x, c)\|_{L^\infty(\mathbb{R}_x^n)L_w^2(\mathbb{R}_c^n)} \leq C(1+t)^{-n/2} + C_N e^{-\epsilon(N)t},$$

with  $w = \exp(-|c|^2/2)$  and  $\epsilon(N) > 0$ .

Here the constant  $C$  is independent of  $N$ , but, in general, we may have asymptotically that  $\epsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ . The validity of the assumption of Theorem 8.7 for all  $N$  is an open problem.

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